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ON CERTAIN SUBGROUPS OF A WREATH PRODUCT

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Abstract: We characterize certain subgroups of a wreath product.

Introduction

Three-dimensional crystallographic space-group symmetry has for a long time been considered the main characteristic of solid-state systems. From the theory of the three-dimensional space groups follow important consequences concerning the physical properties of crystals and one is able to predict such physical properties as energy degeneracies, selection rules and characteristic X-ray diffraction patterns. In recent years however one has become aware of the existence of solid-state systems with long-range order whose symmetry is not a three-dimensional crystallographic space group. Examples of such systems are $NaNO_2$ and K_2MoO_4. Properties of such systems cannot be understood using the theory of the three-dimensional crystallographic space groups.

The description of the symmetry of crystals with structural distortions, developed by Litvin ([4],[5]), is based on the use of a new class of symmetry groups. The method to define such new groups is in turn based on the introduction of new types of operators by coupling additional suitably defined operators to the elements of three-dimensional crystallographic space groups. The mathematical structure of these new groups is related to the concept of wreath products.

Our paper now, shows that the analysis made by Litvin to determine his new groups, can be done entirely in the abstract, mathematical framework of wreath products. So, the inspiration of this paper is "crystallographic", its content is mathematical and we hope it will be useful in crystallography, now that it is shown that Litvin's approach is not restricted to the case considered in [4] .

Preliminaries

If the group G acts as a permutation group on a set Ω and as automorphisms on another group N, then one defines the corresponding wreath product $NWr_\Omega G$ as follows. Let M be the group of all mappings $\Omega \to N$ with pointwise multiplication, then we define $\alpha : G \to Aut(M)$ by $(\alpha_g(m))(\omega) = \beta_g(m(g^{-1}\omega))$, where $g \in G$, $m \in M$, $\omega \in \Omega$; $g^{-1}\omega$ denotes the action of $g^{-1} \in G$ on $\omega \in \Omega$ and β_g denotes the action of $g \in G$ on N. The wreath product $NWr_\Omega G$ is then nothing but the semi-direct product $M \times_\alpha G$, with multiplication: $(m_1,g_1)(m_2,g_2) = (\alpha_{g_2^{-1}}(m_1)m_2 , g_1g_2)$. In this note we will denote the identity element of any group by e.

We should also remark that there exist several different notions of wreath products (cf. [1],[2],[3]). The one we use here has the advantage that one recovers the usual semi-direct product by taking Ω equal to a one-pointset.

Characterization of certain subgroups

1. Definitions

With every element $(m,g) \in NWr_\Omega G$ we associate a bijective mapping $(m,g)^*$ on M, defined by $(m,g)^*(s) = \alpha_{g^{-1}}(s)m$ where $s \in M$. Then $((m,g)^*)^{-1} = ((m,g)^{-1})^*$; $(e,e)^* = id$ and $(m_1,g_1)^*(m_2,g_2)^* = ((m_2,g_2)(m_1,g_1))^*$. So M becomes a right $NWr_\Omega G$-space. We also remark that, identifying M with $M \times \{e\} \subset M \times_\alpha G$, we have:

$$(m,g)^*(s) = (e,g^{-1})(s,e)(m,g) .$$

Finally, we say that a subgroup P of $NWr_\Omega G$ is a stability group for $s \in M$ if $(m,g)^*(s) = s$, for every $(m,g) \in P$.

2. Theorem

For every $s \in M$ and every subgroup D of G, there exists a subgroup P(s,D) of $NWr_\Omega G$, which is a stability group for s and is isomorphic with D.

Proof

With $d \in D$ we associate the element $m_{s,d} \in M$ defined by $m_{s,d} = \alpha_d^{-1}(s^{-1})s$. Then $(m_{s,d} , d)^*(s) = \alpha_d^{-1}(s)\, \alpha_d^{-1}(s^{-1})s = s$. We show that the mapping $f : d \in D \to (m_{s,d} , d) \in NWr_\Omega G$ is an isomorphism on its range. It is obvious that f is bijective, so we only have to show that f is a homomorphism. If $c , d \in D$, then

$$\begin{aligned}(m_{s,c} , c)(m_{s,d} , d) &= (\alpha_c^{-1}(s^{-1})s , c)(\alpha_d^{-1}(s^{-1})s , d)\\ &= (\alpha_d^{-1}(\alpha_c^{-1}(s^{-1})s)\alpha_d^{-1}(s^{-1})s , cd)\\ &= (\alpha_{cd}^{-1}(s^{-1})s , cd)\\ &= (m_{s,cd} , cd) \qquad \text{(Q.E.D.)}\end{aligned}$$

3. Remarks

a) Theorem 2 is actually a result on semi-direct products; it does not use the fact that M is a group of mappings from Ω to N. It can even be formulated for topological groups.

b) $P(e,d) = \{e\} \times D$.

c) If the action of D on M is free (i.e. $\alpha_d(m) = m$ for some $m \neq e$, implies $d = e$), then all the elements $m_{s,d}$ are different.

In the other direction we have the following result.

4. Theorem

Let D be a subgroup of G, with the property that, given $\omega_0 \in \Omega$, every $\omega \in \Omega$ can be written in a unique way in the form $\omega = d\omega_0$, where $d \in D$. Let further P be a subgroup of $NWr_\Omega G$, isomorphic with D, consisting of elements of the form (m_d, d), where $d \in D$, then there exists $s \in M$, such that P is a stability group for s.

Proof

As $(m_d,d)(m_c,c) = (\alpha_c^{-1}(m_d)m_c\ , dc)$, where $c, d \in D$, we see that: $m_{dc} = \alpha_c^{-1}(m_d)m_c$. Choose now $\omega_0 \in \Omega$ and define $s(\omega_0)$ as you wish. Put then $s(\omega) = s(d\omega_0) = \beta_d(s(\omega_0)(m_d(\omega_0))^{-1})$. We will show that $(m_d,d)^*(s) = s$.

We know that $m_{dc}(\omega_0) = (\alpha_c^{-1}(m_d))(\omega_0)m_c(\omega_0)$, hence:

$$\beta_{dc}(s(\omega_0)(m_{dc}(\omega_0))^{-1}\ \beta_{dc}(\alpha_c^{-1}(m_d)(\omega_0)) = \beta_{dc}(s(\omega_0)(m_c(\omega_0))^{-1})$$

$$\Rightarrow\ s(dc\omega_0)\ \beta_d(m_d(c\omega_0)) = \beta_d(s(c\omega_0))$$

$$\Rightarrow\ \forall\omega \in \Omega :\ s(d\omega)\ \beta_d(m_d(\omega)) = \beta_d(s(\omega))$$

$$\Rightarrow\ \forall\omega \in \Omega :\ \beta_d^{-1}(s(d\omega))m_d(\omega) = s(\omega)$$

$$\Rightarrow\ \forall\omega \in \Omega :\ (\alpha_d^{-1}(s))(\omega)m_d(\omega) = s(\omega)$$

or: $$(m_d,d)^*(s) = s \qquad \text{(Q.E.D.)}$$

5. Remark

Theorem 2 and Theorem 4 are generalizations of results of D. Litvin ([4]). He proves the two theorems in the case β is trivial, D = G, N is the three-dimensional euclidean space $\mathbb{R}^3$, considered as an abelian group and G is the symmetry group of a crystal; this is a subgroup of the group of all proper and improper rotations and all translations of $\mathbb{R}^3$ (denoted $\mathcal{E}_s(3)$ in his paper). He then defines a wreath group as a stability group for some "spin arrangement". For more details and applications we refer to his papers ([4] , [5]).

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