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BRAVAIS GROUPS IN LOW DIMENSIONS

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Abstract: Algorithmic methods to compute Bravais groups are described. Full sets of representatives of the $\mathbb{Z}$-classes of all Bravais groups of degree 5 and of the irreducible Bravais groups of degree 6 are given.

I. Introduction.

A Bravais group B is a finite unimodular group (abbreviated f.u. group) for which there exists a set $S \subseteq \mathbb{R}^{n\times n}$ of symmetric matrices (which are interpreted as quadratic forms) such that

$$B = B(S) := \{g \in GL(n,\mathbb{Z}) \mid g^{tr}Xg = X \quad \text{for all} \quad x \in S\}.$$

(The transposed of a matrix g is denoted by g^{tr}.)
A Bravais group can be viewed as the full automorphism group of a suitable lattice in Euclidean n-space since the set S mentioned above can be assumed to consist of one positive definite form (cf. [NPW 80]). Dually to the definition of the Bravais group $B(S)$ of a set of symmetric matrices S the space of quadratic forms fixed by a f.u. group $G(\leq GL(n,\mathbb{Z}))$ is defined by

$$S(G) := \{S \in \mathbb{R}^{n\times n} \mid S^{tr} = S,\ g^{tr}Sg = S \quad \text{for all} \quad g \in G\} .$$

Finally the Bravais group $B(G)$ of the f.u. group G is defined by $B(G) := B(S(G))$. (cf. [BNZ 72,73] Part III, [BBNWZ 78]). Clearly the following holds: $G \leq B(G)$, $B(B(G)) = B(G)$ and $B(G)$ is finite for each f.u. group G .
In [Ple 77] I have described the Bravais group of a given reducible f.u. group. It turned out that for each reducible f.u. group G the "reduction pattern", i.e. the number and degrees of the $\mathbb{Q}$-irreducible constituents and the binding systems are the same for G and $B(G)$. In Chapter II of this note this description of Bravais groups is refined in such a way that one can outline an algorithmic procedure to obtain all reducible Bravais groups in a given dimension once certain irreducible f.u. groups in all lower dimension are known.

The procedure also simplifies the recognition problem considerably, namely to identify the $\mathbb{Z}$-class of the Bravais group of any reducible f.u. group in a list obtained by the suggested algorithm.

The method is applied to the five dimensional case. A complete list of a set of representatives of the $\mathbb{Z}$-classes of the Bravais groups of degree 5 can be found on the attached microfiche. A summary and explanations are given in Chapter III. The irreducible Bravais groups of degree 5 are taken from [PlP 78,80] Part I (cf. also [Bül 73] and [Rys 72]). Lists of Bravais groups of degree smaller than 5 are contained in [BBNWZ 78]. In [PlP 77,80] Part I-V the absolutely irreducible (i.e. $\mathbb{C}$-irreducible) maximal finite subgroups of $GL(n,\mathbb{Z})$ for $5 \leq n \leq 9$ are determined up to $\mathbb{Z}$-equivalence. These are also the absolutely irreducible Bravais groups in these dimensions. As already noted in [PlP 77,80] Part I irreducible groups in prime dimensions are already absolutely irreducible. The same holds for the nine dimensional case. Hence for n smaller than 10 only $n = 6$ and $n = 8$ are the only dimensions for which the irreducible Bravais groups are not known. Therefore we derive the irreducible, not absolutely irreducible Bravais groups of degree 6 in Chapter IV. None of these Bravais groups are maximal irreducible subgroups of $GL(6,\mathbb{Z})$. With the results of Chapter II it should be possible to compute the missing Bravais groups of degree six and seven.

II. The reducible Bravais groups of a fixed dimension.

Among the reducible f.u. groups the fully decomposable ones are the simplest examples. Therefore it is desirable to relate an arbritrary reducible f.u. group to a "canonical" fully decomposed f.u. group which is rationally equivalent to the original group. However, this does not seem to be possible in general. The "almost decomposable" f.u. groups might serve as a substitute.

(II.1) Definition. A finite subgroup G of $GL(n,\mathbb{Z})$ is called almost decomposable, if for each character χ afforded by an

irreducible rational representation of G *the matrix*

$$e_\chi := \frac{\chi^o(1)}{|G|} \sum_{g \in G} \chi(g^{-1})g$$

is integral, where χ^o *is some* $\mathbb{C}$*-irreducible constituent of* χ .

Note that e_χ is an idempotent in $\mathbb{Q}^{n\times n}$ commuting with the elements of G ([Isa 76] pg. 19). Clearly e_χ is zero if χ is not a constituent of the character of the natural representation $\Delta : G \to GL(n,\mathbb{Z}) : g \to g$ of G.
If G is almost decomposable the natural G-lattice $L = \mathbb{Z}^{n\times 1}$ splits into the direct sum of the G-lattices $e_\chi L$ with χ running through the set $Irr(\Delta)$ of the characters of the ($\mathbb{Q}$-)irreducible constituents of Δ :

$$L = \bigoplus_{\chi \in Irr(\Delta)} e_\chi L \ .$$

By choosing $\mathbb{Z}$-bases for each of the $e_\chi L$ and joining them together to a $\mathbb{Z}$-basis of L , one sees that G is $\mathbb{Z}$-equivalent to a group H of block diagonal matrices $diag(h_1,\ldots,h_l)$ where $l = |Irr(\Delta)|$ and the block degrees are multiples $n_\chi \chi(1)$ of $\chi(1)$ for $\chi \in Irr(\Delta)$. The constituent group of the matrices in the j-th block is irreducible if the corresponding n_χ is equal to 1. As pointed out in [BNZ 72,73] Part I the most difficult task for computing reducible f.u. groups is to derive the $\mathbb{Z}$-classes of those constituent groups for which n_χ is bigger than 1. It should be noted, however, that for degrees $\leq$ 7 such a constituent group must already be fully decomposable except if it is isomorphic to D_8 (dihedral group of order eight), A_4 (alternating group on 4 elements) or $C_2 \times A_4$ (cf. [Ple 78]). These are also the cases in which almost decomposable f.u. groups might not be fully decomposable (for degree $\leq$ 7).

Being almost decomposable is a property of the $\mathbb{Z}$-class of a f.u. group in the sense that each f.u. group in a $\mathbb{Z}$-class is almost decomposable if one group of the $\mathbb{Z}$-class has this property.

Within a $\mathbb{Q}$-class of a f.u. group there might be more than one $\mathbb{Z}$-classes of almost decomposable f.u. groups. However the preceeding discussion suggests to associate with any f.u. group G the uniquely defined $\mathbb{Z}$-class of f.u. groups belonging to the action of G on $\tilde{L} = \bigoplus_{\chi \in \mathrm{Irr}(\Delta)} e_\chi L$ $(L = \mathbb{Z}^{n\times 1})$, i.e. the $\mathbb{Z}$-class of all $X^{-1}GX$ where X is the matrix of the basis transformation of the standard basis of L to some basis of $\tilde{L}$. This $\mathbb{Z}$-class is denoted by $\tilde{G}$. As for the Bravais groups, it is now possible to associate a $\mathbb{Z}$-class of almost decomposable Bravais groups with any Bravais group.

(II.2) Definition. *Let $B \leq GL(n,\mathbb{Z})$ be a Bravais group. By $P(B)$ we denote the $\mathbb{Z}$-class of all almost decomposable Bravais groups $B(H)$, where H belongs to the $\mathbb{Z}$-class $\tilde{B}$ (defined above).*

The definition of almost decomposable Bravais groups associated with arbitrary Bravais groups is connected with the intuitive notion of primitivity of lattices in cristallography. For convenience let $P(B)$ also denote some fixed representant of the $\mathbb{Z}$-class $P(B)$. Clearly B is rationally equivalent to a subgroup of $P(B)$. That this might be a proper subgroup can already be seen in three dimensions where a Bravais group B of order 12 exists such that $P(B)$ has order 24. Note that B and $P(B)$ always belong to the same crystal family (cf. [NPW 80]). If the natural representation of B is irreducible, then $B \in P(B)$. In low dimensions it is easy to write down a list of the $\mathbb{Z}$-classes of the almost decomposable Bravais groups. There are only 1,4,8,31, resp. 51 almost decomposable Bravais groups up to $\mathbb{Z}$-equivalence in 1,2,3,4, resp. 5 dimensions, whereas there are 1,5,14,64 resp. 189 $\mathbb{Z}$-classes of Bravais groups falling into 1,4,6,23 resp. 32 crystal families (cf. also [Jar 79] for numbers of crystal families).

The aim of this chapter is to describe for all reducible almost decomposable Bravais group P the set of $\mathbb{Z}$-classes of all Bravais groups B with $P \in P(B)$. To this end let P be a reducible almost decomposable Bravais group and denote by $\mathrm{Min}(P)$ the set of all subgroups H of P with

(i) $B(H) = P$

(ii) for each proper subgroup G of H the Bravais group of G is properly contained in P, i.e. $B(G) \neq P$.

(In the terminology of [BBNWZ 78] $Min(P)$ consists of the minimal subgroups of P belonging to the Bravais flock of P.) Let $L = \mathbb{Z}^{n\times 1}$ denote the natural lattice of P and Δ the natural representation. The canonical decomposition

$$L = \bigoplus_{\chi \in Irr(\Delta)} e_\chi L$$

of L is the same for P and all the groups in $Min(P)$. Consider the set $Cen_P(L)$ of all subgroups L' of L for which the following holds:

(i) There exists a group $H \in Min(P)$ such that L' is an H-sublattice of L, i.e. $HL' = L'$.

(ii) $e_\chi L' = e_\chi L$ for all $\chi \in Irr(\Delta)$.

Note that $Cen_P(L)$ is a finite set, indeed by the definition of e_χ one has $|P|L < L' \leq L$ for all $L' \in Cen_P(L)$. The point about introducing $Cen_P(L)$ is that it is a relatively easy computable set yielding all Bravais groups B with $P \in P(B)$. Namely let the columns of $X \in \mathbb{Z}^{n\times n}$ form a basis of $L' \in Cen_P(L)$. Then $P_X := X^{-1} PX \cap GL(n,\mathbb{Z})$ is a Bravais group with $P \in P(P_X)$. The proof is immediate from [Ple 77]: Because of the definition of $Cen_P(L)$ one has $B(XP_X X^{-1}) = P$, since $XP_X X^{-1}$ is the biggest subgroup of P leaving L' invariant. From [Ple 77] one gets $L = \bigoplus_{\psi \in Irr(\Delta')} e_\psi L'$ where Δ' is the natural representation of the Bravais group $B(P_X)$ of P_X. Hence $B(P_X)$ also leaves L invariant. But P_X is the biggest f.u. group leaving $X^{tr}S(P)X = S(P_X)$ and L invariant, i.e. $B(P_X) = P_X$. Now $P \in P(P_X)$ is clear.

Hence one obtains a mapping of $Cen_P(L)$ into the set of all Bravais $\mathbb{Z}$-classes of Bravais groups B with $P \in P(B)$. This mapping is surjective by definition of $P(B)$ and $Cen_P(L)$. Unfortunately the mapping is not always injective, i.e. there might be several lattices in $Cen_P(L)$ giving rise to the same $\mathbb{Z}$-class of Bravais groups.

This is because the normalizer $N_{GL(n,\mathbb{Z})}(P)$ of P in $GL(n,\mathbb{Z})$ acts on $Cen_P(L)$ via $gL' = \{gl \mid l \in L'\}$ for all $g \in N_{GL(n,\mathbb{Z})}(P)$ and all $L' \in Cen_P(L)$, such that two lattices in the same orbit give rise to the same $\mathbb{Z}$-class of Bravais groups. This follows immediately from the characterization of normalizers of Bravais groups given in [BNZ 72,73] Part III:

$$N_{GL(n,\mathbb{Z})}(P) = \{ g \in GL(n,\mathbb{Z}) \mid g^{tr}S(P)g = S(P) \}.$$

Moreover two different orbits yield different $\mathbb{Z}$-classes of Bravais groups, as will be shown in a moment.

<u>*(II.3) Theorem*</u>. *Let P be an almost decomposable (reducible) Bravais group with natural lattice $L = \mathbb{Z}^{n\times 1}$. There is a 1-1-correspondence between the $\mathbb{Z}$-classes of Bravais groups B with $P \in P(B)$ and the orbits of $Cen_P(L)$ under the action of the normalizer $N_{GL(n,\mathbb{Z})}(P)$ of P in $GL(n,\mathbb{Z})$.*

Proof: All that remains to be proved is that two lattices of $Cen_P(L)$ in different orbits under the action of the normalizer define different $\mathbb{Z}$-classes of Bravais groups. Let L_1 and L_2 be two lattices in $Cen_P(L)$ giving rise to the same $\mathbb{Z}$-class of Bravais groups. Define subgroups $H_i = \{ g \in P \mid gL_i = L_i \}$ for $i = 1,2$ of P. There exist bases of L_1 and L_2 such that $P_{X_1} = X_1^{-1}H_1X_1 = X_2^{-1}H_2X_2 = P_{X_2}$ where the columns of $X_i \in \mathbb{Z}^{n\times n}$ form these bases of L_i for $i = 1,2$. Let $h = X_2X_1^{-1}$. The proof is complete if one shows $hL_1 = L_2$ and $h \in N_{GL(n,\mathbb{Z})}(P)$. But the first claim follows immediately from $L_i = X_iL$ for $i = 1,2$. As for the second statement one sees immediately $h^{tr}S(P)h = S(P)$. Hence by the characterization of $N_{GL(n,\mathbb{Z})}(P)$ quoted before the statement of the theorem $hL = L$ remains to be proved. But $L = \bigoplus_{\chi\in Irr(\Delta)} e_\chi L_i$ for $i = 1,2$, where Δ is the natural representation of P. On the other hand, since $B(H_1) = B(H_2) = P$, it is clear from earlier remarks that

$\{e_\chi \mid \chi \in \mathrm{Irr}(\Delta)\} = \{e_\chi \mid \chi \in \mathrm{Irr}(\Delta_i)\}$ for $i = 1,2$ where Δ_i is the natural representation of H_i $(i=1,2)$. Since h conjugates H_2 to H_1, it also conjugates the idempotents e_χ with $\chi \in \mathrm{Irr}(\Delta_2)$ into the e_χ with $\chi \in \mathrm{Irr}(\Delta_1)$. Hence

$$hL = h \bigoplus_{\chi\in \mathrm{Irr}(\Delta_1)} e_\chi L_1 = \bigoplus_{\chi\in \mathrm{Irr}(\Delta_1)} he_\chi h^{-1} hL_1 = \bigoplus_{\chi\in \mathrm{Irr}(\Delta_2)} e_\chi L_2 = L .$$

q.e.d.

The computation of the $\mathbb{Z}$-classes of reducible Bravais groups in a given dimension n as suggested by Theorem (II.3) proceeds in four steps:

(i) *Finding a set S of representatives of the $\mathbb{Z}$-classes of the almost decomposable (reducible) Bravais groups.*

(ii) *Computing $Cen_P(L)$ (L = natural lattice of P) for each $P \in S$, i.e. by*

a) Finding $Min(P)$

b) Computing the H-sublattices L' of L with $e_\chi L' = e_\chi L$ for all $\chi \in Irr(\Delta)$, where Δ is the natural representation of P.

(iii) *Computing the orbits of $Cen_P(L)$ under $N_{GL(n,\mathbb{Z})}(P)$.*

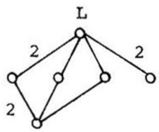
(iv) *For each representative lattice L' of the orbits in $Cen_P(L)$ compute $H(L') = \{h \in P \mid hL' = L'\}$. Then $X^{-1}H(L')X$ is the associated Bravais group, where the columns of $X \in \mathbb{Z}^{n\times n}$ form a $\mathbb{Z}$-basis of L'.*

As mentioned earlier eight is the first critical dimension as far as (i) is concerned. Step (ii)a usually is a simple exercise in elementary group theory. Actually it suffices to compute the groups in $\mathrm{Min}(P)$ up to $\mathbb{Z}$-equivalence, because two $\mathbb{Z}$-equivalent groups of $\mathrm{Min}(P)$ are already conjugate in $N_{GL(n,\mathbb{Z})}(P)$. (ii)a is the heart of the matter. By a slight variation of the centering algorithm developed in [Ple 74] (cf. also [PlP 77,80] Part I) these sublattices can be computed in a very efficient way on a computer.
Step (iii) might cause some difficulties, since $N_{GL(n,\mathbb{Z})}(P)$ is not

always finite. However $Cen_P(L)$ is finite. Only very rarely one actually needs a set of generators for $N_{GL(n,\mathbb{Z})}(P)$. In this case the reader is referred to [BNZ 72,73] Part III. Finally step (iv) is straightforeward again. However the index of $H(L')$ in P might be bigger than one is used from dimensions 3 and 4. For instance in [Ple 78] (Theorem V.3.) a maximal finite reducible subgroup B of $GL(6,\mathbb{Z})$ is constructed, such that the 3-dimensional constituent groups are not maximal finite subgroups of $GL(3,\mathbb{Z})$ and no Bravais groups. In this example the group orders are $|B| = |H(L')| = 2^6 \cdot 3$ and $|P| = 2^8 \cdot 3^2$.

An easy example might serve as an illustration.

<u>*(II.4) Example:*</u> *$P = \{diag(a_1,a_2,a_3) \mid a_i = \pm 1\}$ is an almost decomposable Bravais group of degree 3 with $L = \mathbb{Z}^{3\times 1}$ as natural lattice. The normalizer $N = N_{GL(3,\mathbb{Z})}(P)$ is the full monomial group of order 48. $Min(P)$ consists of the N-conjugates of $\langle diag(-1,1,1),\ diag(1,-1,1)\rangle$ and $\langle diag(1,-1,-1),\ diag(-1,1,-1)\rangle$. $Cen_P(L)$ consist of six lattices, all of which contain $2L$. They fall into four orbits of length 1,3,1,1 under the action of N. All four Bravais groups are in the $\mathbb{Q}$-class of P.*



Theorem (II.3) does not only simplify the derivation of Bravais groups but also the recognition problem. Assume $\mathcal{L}$ is a list of representatives of the $\mathbb{Z}$-classes of Bravais groups in dimension n, which has been produced by the method described above. Let G be some f.u. group of degree n. The problem is to decide which of the groups in $\mathcal{L}$ is $\mathbb{Z}$-equivalent to the Bravais group of G. One first has to decide which of the groups in $\mathcal{L}$ lie in $P(B(G))$, i.e. is in the $\mathbb{Z}$-class of the associated almost decomposable Bravais groups. Then one has to identify the orbit of the natural lattice of G under the action of the normalizer of some representative of the associated almost decomposable Bravais groups.

III. The Bravais groups of degree 5.

There are 32 crystal families in five dimensional space (cf. also [Jar 79]). Let G be some f.u. group of degree n and $\Delta : G \to GL(n,\mathbb{Z}) : g \to g$ the natural representation of G. Then Δ is rationally equivalent to a fully decomposed representation:

$$\Delta \sim_{\mathbb{Q}} \bigoplus_{i=1}^{s} a_i \Delta_i$$

where $a_i \in \mathbb{N}$ and the Δ_i are rational irreducible representations $(1 \leq i \leq s)$, no two of which are rationally equivalent. Let n_i be the degree of Δ_i. The Δ_i can be ordered in such a way that $n_1 \geq n_2 \geq \ldots \geq n_s$ and whenever $n_i = n_{i+1}$ the inequality $a_i \geq a_{i+1}$ holds. The $(a_1+\ldots+a_s)$-tuple

$$(\underbrace{\overline{n_1,\ldots,n_1}}_{a_1}, \underbrace{\overline{n_2,\ldots,n_2}}_{a_2},\ldots,\underbrace{\overline{n_s,\ldots,n_s}}_{a_s})$$

will be called the decomposition scheme of G, where the bars above the n_i's are omitted if $a_i = 1$ $(1 \leq i \leq s)$. For instance an irreducible group of degree n has (n) as decomposition scheme; $(\overline{1,1,1})$, $(\overline{1,1},1)$, $(2,1)$, (3) are the possible decomposition schemes of f.u. groups of degree 3. It follows immediately from [Ple 77] or [Jar 79] that the decomposition scheme is a family invariant, i.e. all f.u. groups within the same crystal family have the same decomposition scheme. The second important family invariant is the dimension of the space of quadratic form fixed by some group in the crystal family.

The following table gives a systematic account on the distribution of $\mathbb{Z}$-classes of Bravais groups into crystal families for the five dimensional space. The first column contains the number of the family starting with those families having form spaces of highest dimension. These dimensions are listed in the third column (number of parameters). The second column gives the decomposition scheme defined above. In the fifth and sixth column (number of $\mathbb{Z}$-classes of a.d. Bravais groups, isom. types of a.d. Bravais groups) the number of $\mathbb{Z}$-classes and the isomorphism types of the almost decomposable Bravais groups in the corresponding family are given. The following notation is used: C_n cyclic group of order n, D_{2n} dihedral group of order 2n, S_n symmetric group on n elements, $G \times H$ direct product of G and H, $G \sim S_n$ wreath product of G by S_n of order $|G|^n \cdot n!$, G^n direct product of n copies of G , $W(F_4)$ Weylgroup of root system F_4. Finally the last column gives the number of $\mathbb{Z}$-classes of Bravais groups in the family. If there is more than one $\mathbb{Z}$-class of almost decomposable Bravais groups in the family, the entry in the last column is given in the form $r_1 + \ldots + r_s$ where r_i $(1 \leq i \leq s)$ is the number of $\mathbb{Z}$-classes of Bravais groups associated with the ith $\mathbb{Z}$-class of almost decomposable Bravais groups as described in chapter II.

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Family number	decomp. scheme	number of para-meters	numb. of $\mathbb{Z}$-cl. of a.d.Br.gr.	isom.types of a.d.Bravais gr.	number of $\mathbb{Z}$-classes of Bravais groups
I	$(\overline{1,1,1,1,1})$	15	1	C_2	1
II	$(\overline{1,1,1,1},1)$	11	1	$C_2{}^2$	2
III	$(\overline{1,1,1},\overline{1,1})$	9	1	$C_2{}^2$	3
IV	$(\overline{1,1,1},1,1)$	8	1	$C_2{}^3$	6
V	$(\overline{1,1},\overline{1,1},1)$	7	1	$C_2{}^3$	9
VI	$(2,\overline{1,1,1})$	7	1	$D_8 \times C_2$	2
VII	$(2,\overline{1,1,1})$	7	1	$D_{12} \times C_2$	2
VIII	$(\overline{1,1},1,1,1)$	6	1	$C_2{}^4$	18
IX	$(1,1,1,1,1)$	5	1	$C_2{}^5$	19
X	$(2,\overline{1,1},1)$	5	1	$D_8 \times C_2{}^2$	8
XI	$(2,\overline{1,1},1)$	5	1	$D_{12} \times C_2{}^2$	6
XII	$(\overline{2,2},1)$	5	1	$C_4 \times C_2$	2
XIII	$(\overline{2,2},1)$	5	1	$C_6 \times C_2$	2
XIV	$(2,1,1,1)$	4	1	$D_8 \times C_2{}^3$	15
XV	$(2,1,1,1)$	4	1	$D_{12} \times C_2{}^3$	9
XVI	$(\overline{2,2},1)$	4	3	$D_8 \times C_2$	2+3+2
XVII	$(\overline{2,2},1)$	4	2	$D_{12} \times C_2$	2+2
XVIII	$(3,\overline{1,1})$	4	3	$(C_2 \sim S_3) \times C_2$	3+2+1
XIX	$(2,2,1)$	3	1	$D_8{}^2 \times C_2$	8
XX	$(2,2,1)$	3	1	$D_8 \times D_{12} \times C_2$	4
XXI	$(2,2,1)$	3	1	$D_{12}{}^2 \times C_2$	7
XXII	$(3,1,1)$	3	3	$(C_2 \sim S_3) \times C_2{}^2$	8+5+2
XXIII	$(4,1)$	3	1	$D_{16} \times C_2$	2
XXIV	$(4,1)$	3	1	$D_{24} \times C_2$	1
XXV	$(4,1)$	3	1	$D_{20} \times C_2$	2

5-DIMENSIONAL BRAVAIS GROUPS

Family number	decomp. scheme	number of parameters	numb. of $\mathbb{Z}$-cl. of a.d.Br.gr.	isom.types of a.d.Bravais gr.	number of $\mathbb{Z}$-classes of Bravais groups
XXVI	(3,2)	2	3	$(C_2 \sim S_3) \times D_8$	2+2+1
XXVII	(3,2)	2	3	$(C_2 \sim S_3) \times D_{12}$	1+2+2
XXVIII	(4,1)	2	2	$(C_2 \sim S_4) \times C_2$, $W(F_4) \times C_2$	2+2
XXIX	(4,1)	2	2	$(D_{12} \sim S_2) \times C_2$, $(C_2 \times (D_6 \sim S_2)) \times C_2$	2+2
XXX	(4,1)	2	2	$(C_2 \times S_5) \times C_2$	3+1
XXXI	(5)	1	3	$C_2 \sim S_5$	1+1+1
XXXII	(5)	1	4	$C_2 \times S_6$	1+1+1+1

The table can be used as a guide for reading the attached microfiche. The Bravais groups are listed there in the same order as in the table. for each family the following information is provided by the microfiche:

a) The form space of a "natural" representative P the $\mathbb{Z}$-classes of almost decomposable Bravais groups of the family.
b) For each P under a) the order $|P|$ of P.
c) For each P under a) a set of representatives $L_1,\dots,L_{r(P)}$ of the orbits of $\mathrm{Cen}_P(\mathbb{Z}^{5\times 1})$ under the action of $N_{GL(n,\mathbb{Z})}(P)$ in the form of matrices $X_1,\dots,X_{r(P)} \in \mathbb{Z}^{5\times 5}$ such that the columns of X_i form a basis of L_i $(1\leq i\leq r(P))$. (The r(P) are listed in the last column of the above table.)
d) For each L_i under c) a set of generators of $H(P,i) = \{g\in P \mid gL_i = L_i\}$ and the corresponding generators of the Bravais group $P_{X_i} = X_i^{-1} H(P,i)X_i$ associated with L_i. (These P_{X_i} form a set of representatives of the $\mathbb{Z}$-classes of Bravais groups of degree 5.)
e) For each L_i under c) the index of $H(P,i)$ in P.

IV. The irreducible Bravais groups of degree six.

The maximal finite absolutely irreducible subgroups of $GL(6,\mathbb{Z})$ were determined in [PlP 77,80] Part II. They are the absolutely irreducible Bravais groups of degree 6 and fall into 20 $\mathbb{Z}$-classes. Therefore only the irreducible, not absolutely irreducible Bravais groups of degree 6 are discussed in this chapter. The following well known lemma restricts the possible dimensions of the form spaces.

(VI.1) Lemma. Let G be an f.u. group and χ *the character of the natural representation of G. Let*

$$\chi = \sum_{i=1}^{r} a_i\chi_i + 2\sum_{i=1}^{q} b_i\psi_i + \sum_{i=1}^{k} c_i(\vartheta_i+\bar{\vartheta}_i)$$

be the decomposition of χ *into irreducible complex characters such the* χ_i *,* ψ_i *and* $\vartheta_i,\bar{\vartheta}_i$ *are irreducible and pairwise different, the* χ_i

are afforded by real representations ($1 \le i \le r$), *the* ψ_i *are real, but not afforded by real representation* ($1 \le i \le q$), *and the* ϑ_i *are not real, i.e.* $\vartheta_i \neq \overline{\vartheta}_i$, *where* $\overline{\vartheta}_i$ *is the complex conjugate of* ϑ_i ($1 \le i \le k$). *Then the dimension of the space of forms* $S(G)$ *is given by*

$$\frac{1}{2} \sum_{i=1}^{r} (a_i^2 + a_i) + \sum_{i=1}^{q} (2b_i^2 - b_i) + \sum_{i=1}^{k} c_i^2 \quad .$$

Proof: Let (,) denote the scalar product for class functions of G, and 1 the 1-character of G. For each character α of G let $\alpha^{(2)}$ denote the class function defined by $\alpha^{(2)}(g) = \alpha(g^2)$ for $g \in G$. Then the action of G on the symmetric matrices $\left(g \to \begin{pmatrix} X \\ g^{tr}Xg \end{pmatrix}, X^{tr} = X\right)$ has the character $\frac{1}{2}(\chi^2 + \chi^{(2)})$. Hence $\dim S(G) = (\frac{1}{2}(\chi^2 + \chi^{(2)}), 1) =$ $= \frac{1}{2}(\chi^2 + \chi^{(2)}, 1)$. But $(\chi^2, 1) = (\chi, \overline{\chi}) = (\chi, \chi) = \sum_{i=1}^{r} a_i^2 + 4 \sum_{i=1}^{q} b_i^2 + 2 \sum_{i=1}^{k} c_i^2$. From the results by Frobenius and Schur (cf. [Isa 76] pg. 49 ff.) it is well known that $(\chi_i^{(2)}, 1) = 1$ for $1 \le i \le r$, $(\psi_i^{(2)}, 1) = -1$ for $1 \le i \le q$, and $(\vartheta_i^{(2)}, 1) = (\overline{\vartheta}_i^{(2)}, 1) = 0$ for $1 \le i \le k$. From this the result follows immediately.

q.e.d.

Now let G be an irreducible, not absolutely irreducible f.u.group of degree six and let χ be the character of the natural representation of G. Let

$$\chi = \sum_{i=1}^{l} a_i \chi_i$$

be the decomposition of χ in irreducible complex characters. Then the χ_i $(1 \le i \le l)$ are algebraic conjugate and hence have the same degree, say m. Also the a_i $(1 \le i \le l)$ are equal, say $a_i = a$. Then $l \cdot a \cdot m = 6$. This equation has the following solutions:

number	i	ii	iii	iv	v	vi	vii	viii	ix
l	1	1	1	1	2	2	3	3	6
a	1	2	3	6	1	3	1	2	1
m	6	3	2	1	3	1	2	1	1

In the case of solution (i) G is absolutely irreducible. Since a is the Schurindex of χ_1 it divides the degree m of χ_1 (cf. [Isa 76] p.161). Therefore there does not exist a group in the cases (ii),(iii),(iv),(vi), or (viii). Only the cases (v),(vii), and (ix) are to be considered. All three cases are discussed seperately.
Case (v):
(a) χ_1 is not real. In this case $\dim S(G) = 1$ and G is isomorphic to a subgroup $\tilde{G}$ of $GL(3,K)$, where $K = \mathbb{Q}(\chi_1)$ is a complex quadratic extension of $\mathbb{Q}$. (This can be seen by reducing the natural representation of G over K.) In [Bli 17] all finite irreducible subgroups of $GL(3,\mathbb{C})$ are described. They either contain an irreducible imprimitive normal subgroup or they contain a normal subgroup isomorphic to A_5, to a central extension $\tilde{A}_6$ by C_3 or to $PSL(2,7)$. Note that the minimal splitting field of the irreducible representation of A_5 of degree 3 is real quadratic, namely $\mathbb{Q}[\sqrt{5}]$. Hence it need not to be considered in this case. The matrix group isomorphic to $\tilde{A}_6$ has a biquadratic extension of $\mathbb{Q}$ as minimal splitting field and therefore need not be considered at all. Finally the group isomorphic to $PSL(2,7)$ contains an irreducible imprimitive subgroup isomorphic to an extension of C_7 by C_3.

(IV.2) Lemma. There is no irreducible Bravais group B of degree 6, such that the character χ of the natural representation of B is the sum of two complex conjugate ($\mathbb{C}$-) irreducible characters.

Proof: The statement of the lemma is equivalent to the following: For each irreducible f.u. group G of degree six with the properties (i) and (ii) $B(G)$ is absolutely irreducible, where

(i) the character of the natural representation of G is the sum of two complex conjugate ($\mathbb{C}$-) irreducible characters.

(ii) G has no proper subgroup with property (i).
By the remarks preceeding (IV.2), G is isomorphic to an imprimitive subgroup G of $GL(3,\mathbb{C})$, which is irreducible but does not contain any proper irreducible subgroups. All relevant groups are conjugate

in $GL(3,\mathbb{C})$ to $\left\langle \begin{pmatrix} \vartheta & 0 & 0 \\ 0 & \vartheta^2 & 0 \\ 0 & 0 & \vartheta^4 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle$, (order $7 \cdot 3$),

where ϑ is a primitive seventh root of unity, or

to a subgroup of $\left\langle \begin{pmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle$ or

$\left\langle \begin{pmatrix} i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle$ where ω is a primitive sixth and

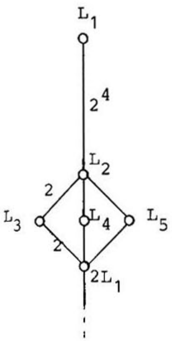
i a primitive fourth root of unity. From this one concludes that there there are six $\mathbb{Q}$-classes of f.u. groups G of degree six satisfying properties (i) and (ii) (orders $7 \cdot 3$, 3^3, 3^4, $2^2 \cdot 3^2$, $4^2 \cdot 3$, $4 \cdot 2^2 \cdot 3$) Electronic computation of the lattices left invariant by representants of the six $\mathbb{Q}$-classes show that the quadratic forms fixed by an f.u. group of degree six with (i) and (ii) above are $\mathbb{Z}$-equivalent to the quadratic forms fixed by the absolutely irreducible subgroups of $GL(6,\mathbb{Z})$ derived in [PlP p.77,80] Part II, which proves the lemma. (The $\mathbb{Q}$-classes of these groups fall into 3,3,6,3,5, resp. 5 $\mathbb{Z}$-classes.The ring of integral matrices commuting with the groups are isomorphic as $\mathbb{Z}$-order to $\mathbb{Z}\left[\frac{-1+\sqrt{-7}}{2}\right]$, $\mathbb{Z}\left[\frac{-1+\sqrt{-3}}{2}\right]$, resp. $\mathbb{Z}[\sqrt{-1}]$, all of which have class number one; compare proof of (IV.3) for the exact method.)

q.e.d.

(b) χ_1 is real. In this case $\dim S(G) = 2$ and it follows from the discussion at the beginning of (a) that G is isomorphic to A_5 or $C_2 \times A_5$.

(IV.3) Lemma. *There are three $\mathbb{Z}$-classes of irreducible Bravais groups B of degree six such that the character χ is the sum of two algebraically conjugate ($\mathbb{C}$-)irreducible real characters. All these groups fall into one $\mathbb{Q}$-class. The isomorphism type is $C_2 \times A_5$ (A_5 = alternating groups on 5 elements).*

Proof: Since $-I_n$ is contained in every Bravais group of degree n, the isomorphism type is $C_2 \times A_5$ and from the character table of $C_2 \times A_5$ it is clear that only one $\mathbb{Q}$-class exists. That all f.u. groups in the $\mathbb{Q}$-class are Bravais groups follows from the remarks preceding (IV.3) and from (IV.1). The centering algorithm (applied to the group B_1 of (IV.5)) yields the following lattice of submodules of 2-power index in $\mathbb{Z}^{6\times 1}$. The lattices L_3, L_4, L_5 are easily seen to be isomorphic. L_1, L_2, L_3 yield a set of representatives, as can be verified by the following:



(i) Because of the submodule lattice, L_1, L_2, and L_3 certainly give rise to three different $\mathbb{Z}$-classes of groups G_1, G_2, G_3 and any lattice contained in L_1 with 2-power index is isomorphic to L_1, L_2, or L_3.

(ii) The centralizer of G_1, G_2, resp. G_3 in $\mathbb{Z}^{6\times 6}$ is isomorphic to $\mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]$, $\mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]$, resp. $\mathbb{Z}[\sqrt{5}]$.

(iii) For i = 1,2 the following holds: Each sublattice of L_i of index prime to 2 is of the form L_i, where is an ideal of the centralizer of G_1 in $\mathbb{Z}^{6\times 6}$. Since $\mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]$ has class number 1, one has $\mathfrak{a} L_i = aL_i$ for a suitable $a \in \mathfrak{a}$, i.e. $\mathfrak{a}L_i \cong L_i$ as $\mathbb{Z}(C_2\times A_5)$-lattices.

(iv) Each sublattice $\tilde{L}$ of L_3 of 2'-index in L_3 is contained in a sublattice of L_2 of index 2. Hence because of (iii) and

$L_3 \cong L_4 \cong L_5$ one has $\tilde{L} \cong L_3$ as $\mathbb{Z}(C_2 \times A_5)$-lattices. (cf. also [Jac 68] for a more general argument.

q.e.d.

Case (vii). By reducing the natural representation Δ of G over the field $K = \mathbb{Q}(\chi_1)$, where χ_1 is an $\mathbb{C}$-irreducible constituent of the character χ of Δ, one sees that G is isomorphic to a subgroup of $GL(2,K)$ ($\subseteq GL(2,\mathbb{C})$). However, the finite complex linear groups of degree two were already determined by F. Klein (cf. [Bli 17]). They are either imprimitive or contain a subgroup isomorphic to the binary tetrahedral, octahedral, or icosahedral group. The latter three groups certainly cannot arise in the situation considered here, since they cannot be embedded into the linear group of degree two over a cubic extension of the rationals. For $[K : \mathbb{Q}] = 3$, since χ_1 has three algebraic conjugates. Since K can be embedded in a cyclotomic field ($|G| < \infty$), K is a Galois extension of $\mathbb{Q}$. Hence K is contained in $\mathbb{R}$, since $[K : \mathbb{Q}]$ is odd. Therefore G is isomorphic to a finite $\mathbb{C}$-irreducible real orthogonal group of degree two, i.e. G is a dihedral group. By Lemma (IV.1) dim $S(G) = 3$.

(IV.4) Lemma. *There are two $\mathbb{Z}$-classes of irreducible Bravais groups B of degree six such that the character χ of the natural representation is the sum of three ($\mathbb{C}$-) irreducible algebraically conjugate characters. The groups are isomorphic to dihedral groups D_{28} and D_{36} of orders 28 and 36.*

Proof: It is clear from the preceding remarks that B is isomorphic to a dihedral group D_{2n} with $\varphi(n) = 6$ where φ is the Euler φ-function. Hence $n \in \{7,14,9,18\}$. But for $n = 7$ or $n = 9$ B does not contain $-I_6$ (I_n = unit matrix of degree n), and therefore cannot be a Bravais group. Clearly there are subgroups of $GL(6,\mathbb{Z})$ isomorphic to D_{28} and D_{36}. They have to be Bravais groups because every f.u. group containing one of them properly is already absolutely irreducible and has only a one dimensional space of quadratic forms by the above discussion and Lemma (IV.1). It remains to show that there is only one

$\mathbb{Z}$-class in each case. B has a unique cyclic subgroup of index 2, which is irreducible. By (IV.1) B is the Bravais group of this subgroup. Hence it suffices to show that there is only one $\mathbb{Z}$-class of cyclic groups of order 14 resp. 18 of f.u. groups of degree 6. But this is clear since the classnumber of the 14th resp. 18th cyclotomic field is one.

q.e.d.

Remark. The preceeding arguments can be extended to prove that in every even dimension n there are irreducible Bravais groups, isomorphic to dihedral groups D_{2k} with k even and $\varphi(k) = n$. The number of $\mathbb{Z}$-classes within a $\mathbb{Z}$-class is one if the k-th cyclotomic field has class number one. The dimension of the space of quadratic forms fixed by the groups is $\frac{n}{2}$. (cf. [BNZ 72,78] Part II, Theorem 7.4 for the 4-dimensional case).

Case (ix)

In this case χ is the sum of 6 linear characters which are algebraically conjugate. Hence G is cyclic. The proof of (IV.4) shows: (i) G has order 7,14,9, or 18, (ii) there is just one $\mathbb{Z}$-class in each case and (iii) the Bravais groups are dihedral groups of order 28 or 36. Hence there is no Bravais group in this case.

The results of this chapter are summarized in the following theorem.

(IV.5) Theorem. Let B be a ($\mathbb{Q}$-)irreducible but not ($\mathbb{C}$-)irreducible Bravais group of degree six. Then B is $\mathbb{Z}$-equivalent to one of the groups B_i for $1 \leq i \leq 5$ listed below. B_1, B_2 and B_3 are $\mathbb{Q}$-equivalent and isomorphic to $C_2 \times A_5$; B_4 and B_5 are dihedral groups of order 28 and 36. The groups and their form spaces are:

$$B_1 = \left\langle a_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 \end{bmatrix},\ b_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & -1 & 0 & 0 & 0 \\ -2 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \end{bmatrix},\ -I_6 \right\rangle$$

$$\text{with} \quad S(B_1) = \{a\begin{pmatrix} 2&1&0&0&0&0\\ 1&2&1&0&0&0\\ 0&1&2&1&0&0\\ 0&0&1&2&1&1\\ 0&0&0&1&2&0\\ 0&0&0&1&0&2 \end{pmatrix} + b\begin{pmatrix} 2&3&0&-2&0&0\\ 3&2&1&-2&-2&-2\\ 0&1&2&-1&0&-4\\ -2&-2&-1&-2&1&-3\\ 0&-2&0&1&2&0\\ 0&-2&-4&-3&0&-2 \end{pmatrix} \mid a,b \in \mathbb{R}\},$$

$$B_2 = \left\langle a_2 = \begin{pmatrix} 0&1&0&0&0&1\\ 1&0&0&0&0&1\\ 0&0&0&-1&0&0\\ 0&0&-1&0&0&0\\ 0&0&0&0&-1&0\\ 0&0&0&0&0&-1 \end{pmatrix},\ b_2 = \begin{pmatrix} 0&1&0&1&0&1\\ 0&0&1&1&0&1\\ 1&0&0&1&0&1\\ 0&0&0&1&1&1\\ 0&0&0&1&0&0\\ 0&0&0&-2&0&-1 \end{pmatrix},\ -I_6 \right\rangle$$

$$\text{with} \quad S(B_2) = \{a\begin{pmatrix} 2&0&0&0&0&1\\ 0&2&0&0&0&1\\ 0&0&2&0&0&1\\ 0&0&0&2&0&1\\ 0&0&0&0&2&1\\ 1&1&1&1&1&3 \end{pmatrix} + b\begin{pmatrix} 0&2&2&-2&2&3\\ 2&0&2&-2&-2&-1\\ 2&2&0&2&-2&3\\ -2&-2&2&0&-2&-1\\ 2&-2&-2&-2&0&-1\\ 3&-1&3&-1&-1&3 \end{pmatrix} \mid a,b \in \mathbb{R}\},$$

$$B_3 = \left\langle a_3 = \begin{pmatrix} 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&-1&0&0\\ 0&0&-1&0&0&0\\ 0&0&0&0&-1&0\\ 0&0&0&0&0&-1 \end{pmatrix},\ b_3 = \begin{pmatrix} 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&-1\\ 0&0&0&-1&0&0 \end{pmatrix},\ -I_6 \right\rangle$$

$$\text{with} \quad S(B_3) = \{a\, I_6 + b\begin{pmatrix} 0&1&1&-1&1&1\\ 1&0&1&-1&-1&-1\\ 1&1&0&1&-1&1\\ -1&-1&1&0&-1&1\\ 1&-1&-1&-1&0&1\\ 1&-1&1&1&1&0 \end{pmatrix} \mid a,b \in \mathbb{R}\},$$

where $a_i^{\,2} = b_i^{\,3} = (a_i b_i)^5 = I_6$ *holds for* $i = 1,2,3$;

$$B_4 = \left\langle c_1 = \begin{pmatrix} 0&0&0&0&0&1\\ -1&0&0&0&0&1\\ 0&-1&0&0&0&1\\ 0&0&-1&0&0&1\\ 0&0&0&-1&0&1\\ 0&0&0&0&-1&1 \end{pmatrix},\quad d = \begin{pmatrix} 0&0&0&0&0&1\\ 0&0&0&0&1&0\\ 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ 0&1&0&0&0&0\\ 1&0&0&0&0&0 \end{pmatrix} \right\rangle$$

with $c_1^{\,14} = d^2 = (c_1 d)^2 = I_6$ *and*

$$S(B_4) = \{ a \begin{pmatrix} 6 & -1 & -1 & -1 & -1 & -1 \\ -1 & 6 & -1 & -1 & -1 & -1 \\ -1 & -1 & 6 & -1 & -1 & -1 \\ -1 & -1 & -1 & 6 & -1 & -1 \\ -1 & -1 & -1 & -1 & 6 & -1 \\ -1 & -1 & -1 & -1 & -1 & 6 \end{pmatrix} + b \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$+ c \begin{pmatrix} 4 & -1 & -1 & 0 & 0 & -1 \\ -1 & 4 & -1 & -1 & 0 & 0 \\ -1 & -1 & 4 & -1 & -1 & 0 \\ 0 & -1 & -1 & 4 & -1 & -1 \\ 0 & 0 & -1 & -1 & 4 & -1 \\ -1 & 0 & 0 & -1 & -1 & 4 \end{pmatrix} \quad | a,b,c \in \mathbb{R} \},$$

and finally

$$B_5 = \left\langle c_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}, d \right\rangle \quad with$$

$$c_2^{18} = d^2 = (c_2 d)^{18} = I_6 \quad and$$

$$S(B_5) = \{ a \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} + b \begin{pmatrix} 0 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & -1 \\ -1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1 & 0 & 0 \end{pmatrix}$$

$$+ c \begin{pmatrix} 0 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 0 \end{pmatrix} \quad | a,b,c \in \mathbb{R} \}.$$

It should be noted that none of the groups $B_1, B_2, \ldots, B_5$ is a maximal finite subgroup of $GL(6,\mathbb{Z})$. Namely in the terminology of the microfiche supplement of [PlP 77,80] Part V the following inclusions can easily be verified:

$B_1 \subseteq G(2)\,(\cong C_2 \sim S_6)$ and $B_1 \sim_{\mathbb{Z}} \tilde{B}_1 \subseteq G(16)\,(\cong C_2 \times S_5)$;

$B_2 \subseteq G(3)\,(\cong C_2 \sim S_6)$ and $B_2 \sim_{\mathbb{Z}} \tilde{B}_2 \subseteq G(17)\,(\cong C_2 \times S_5)$;

$B_3 \subseteq G(1)\,(\cong C_2 \sim S_6)$ and $B_3 \sim_{\mathbb{Z}} \tilde{B}_3 \subseteq G(15)\,(\cong C_2 \times S_5)$;

$B_4 \subseteq G(12)\,(\cong C_2 \times S_7)$, $B_4 \sim_{\mathbb{Z}} \tilde{B}_4 \subseteq G(13)\,(\cong C_2 \times S_7)$,

and $B_4 \sim_{\mathbb{Z}} \tilde{\tilde{B}}_4 \subseteq G(14)\,(\cong C_2 \times PGL(2,7))$;

$B_5 \subseteq G(7)\,(\cong D_{12} \sim S_3)$, $B_5 \sim_{\mathbb{Z}} \tilde{B}_5 \subseteq G(8)\,(\cong W(E_6) \times C_2)$

and $B_5 \sim_{\mathbb{Z}} \tilde{\tilde{B}}_5 \subseteq G(9)\,(\cong W(E_6) \times C_2)$, where $W(E_6)$ is the Weylgroup of the root system E_6.

One easily checks now that there are seven crystal classes of irreducible f.u. groups, three of which do not contain absolutely irreducible groups. This completes the list of numbers of crystal families for the six-dimensional case in [Jar 79].

References

[Bli 17] H.F. Blichfeldt: Finite Collineation Groups. Chicago 1917.

[BBNWZ 78] H. Brown, R. Bülow, J. Neubüser, H. Wondratschek, H. Zassenhaus: Crystallographic Groups of Four-Dimensional Space. New York 1978.

[BNZ 72,73] H. Brown, J. Neubüser, H. Zassenhaus: On integral groups; I. The reducible case. Num. Math. 19 (1972) 386-399; II. The irreducible case. Num. Math. 20 (1972) 22-31; III. Normalizers. Math. Comput. 27 (1973) 167-182.

[Bül 73] R. Bülow: Über Dadegruppen in $GL(5,\mathbb{Z})$. Dissertation RWTH Aachen 1973.

[Isa 76] I.M. Isaacs: Character Theory of finite groups. New York 1976.

[Jac 68] H. Jacobinski: Genera and Decomposition of Lattices over Orders. Acta Math. 121 (1968) 1-29.

[Jar 79] J.D. Jarrat: The decomposition of crystal families. Univ. of Auckland, New Zealand, Dep. of Math., Report Series No. 151, (1979).

[NPW 80] J. Neubüser, W. Plesken, H. Wondratschek: An emendatory discursion on defining crystal systems. This volume.

[Ple 74] W. Plesken: Beiträge zur Bestimmung der endlichen irreduziblen Untergruppen von GL(n,ℤ). Dissertation RWTH Aachen 1974.

[Ple 77] W. Plesken: The Bravais Group and the Normalizer of a Reducible Finite Subgroup of GL(n,ℤ). Comm. in Algebra, 5(4)(1977) 375-396.

[Ple 78] W. Plesken: On reducible and decomposable representations of orders. Journ. f.d. reine u. angew. Mathem. 297(1978) 188-210.

[PlP 77,80] W. Plesken, M. Pohst: On maximal finite irreducible subgroups of GL(n,ℤ). I. The five and seven dimensional case. II. The six dimensional case. Math. Comput. 31, 138(1977) 536-577. III. The nine dimensional case. IV. Remarks on even dimensions with applications to n = 8. V. The eight dimensional case and a complete description of dimensions less than ten. Math. Comput. 34, 149(1980) to appear.

[Rys 72] Maximal finite groups of integer (n×n)-matrices and full groups of integral automorphisms of positive quadratic forms (Bravais models). Tr. Mat. Inst. Steklov 128 (1972) 183-211. Proc. Steklov Inst. Math. 128 (1972) 217-250.

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