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WEAK EQUIVALENCE OF IRREDUCIBLE REPRESENTATIONS OF LITTLE SPACE GROUPS

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In the quantum mechanical study of a physical system S possessing a symmetry group G, one has a unitary representation of G in the Hilbert space H of state vectors of S, i.e. a homomorphism $f : G \to U(H)$ of G into the unitary group of H. Generally, f has a non-trivial kernel $\mathrm{Ker}\, f$, so its image $\mathrm{Im}\, f = \{f(g)\ ;\ g \in G\} = f(G)$ is isomorphic to the quotient group $G/\mathrm{Ker} f$. One says that G does not act effectively in H and that it acts on the state vectors only through the image $f(G)$. So the physical phenomena of S depend only on $f(G)$. The memory of the kernel $\mathrm{Ker}\, f$ is lost.

Let us note that if the dimension of H is finite then $f(G)$ is just the set of matrices which appear in the representation f of G.

Two n-dimensional images $f_1(G_1)$ and $f_2(G_2)$ are called equivalent (we write $f_1(G_1) \sim f_2(G_2)$) if they are conjugated subgroups in the group $GL(n,\mathbb{C})$ of non-singular $n \times n$ matrices. In other words, $f_1(G_1) \sim f_2(G_2)$ if and only if there exists a matrix

$W \in GL(n,\mathbb{C})$ such that $Wf_1(G_1)W^{-1} = f_2(G_2)$, i.e. the whole image $f_1(G_1)$ is transformed onto the whole image $f_2(G_2)$. Let us note that this equivalence condition is weaker than the usual equivalence of representations when $G_1=G_2=G$, namely the usual equivalence (denoted by $\approx$) means that $Wf_1(g)W^{-1} = f_2(g)$ for every $g \in G$. That is why the relation $\sim$ will be called a weak equivalence.

The weak equivalence relation $\sim$ was proposed in [1] , [2] . The equivalent images of different space group representations have the same invariants, [1] . Such invariants are computed, for example, in Landau's theory of phase transitions.

It is natural to classify the lattice-vibration representations by the weak equivalence, too, [3]. In this context, the equivalence $\sim$ is also motivated by the fact that the polarisation vectors can be related to the eigenvalues of matrices appearing in the corresponding lattice-vibration representation of the group G_k of a wave vector k (little group of k), [4] .

All images $D_k(G_k)$ of allowed irreducible representations D_k of G_k groups (high-symmetry wave vectors k) are listed and discussed in [5],[6] . The images $D_k(G_k)$ have the following properties:

1. $D_k(G_k) = T_k^{(m)}B_k$, where $T_k^{(m)} = \{e^{-ikt};\ t \in T\}$, T is the translation subgroup of G_k, m denotes the order of $T_k^{(m)}$ ($T_k^{(m)}$ is a cyclic group if k is a high-symmetry wave vector) and B_k is a group of $n \times n$ matrices ($n = \dim D_k$)
2. For a few thousands of single-valued allowed irreducible representations of G_k (high-symmetry wave vectors) there are only 25 weak equivalence classes of the B_k groups
3. Every B_k is either a unitary reflection group or is a proper subgroup of such group (unitary reflection groups are defined and discussed, for example, in [7],[8],[9]).

In the context of property 3 one can note that the lattice vibration representation L_k of G_k can be written in the form $L_k = F_k \otimes P_k$ where $\otimes$ denotes the tensor product of group representations, P_k is the vector representation of the point group of G_k, and F_k (defined by the formula (9.16) in [10]) is a subgroup of the unitary reflection group $G(m,p,n)$, [8],[9] , when m denotes the order of $T_k^{(m)}$, and n is the number of atoms in the Wigner--Seitz cell, [3] .

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