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AN EQUIVALENCE TEST FOR MULTIPLIERS OF PROJECTIVE REPRESENTATIONS OF SOLVABLE GROUPS

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1. Introduction

Let G be a discrete group. By a projective unitary representation of G of dimension n we mean a map $D : G \to U(n)$ together with a unit circle-valued function ω on $G \times G$ which satisfies

$$D(g_1)\, D(g_2) = \omega(g_1, g_2)\, D(g_1 g_2), \tag{1}$$

for all $g_1, g_2 \in G$. ω is called a multiplier (or factor system) for G, and, if we wish to emphasize ω, D is said to be an ω-representation. The associative law for group multiplication forces ω to satisfy

$$\omega(g_1, g_2)\, \omega(g_1 g_2, g_3) = \omega(g_1, g_2 g_3)\, \omega(g_2, g_3), \tag{2}$$

for all $g_1, g_2, g_3 \in G$.

If α is any unit circle-valued function on G we can obtain a new projective representation D' by defining $D'(g) = \alpha(g)\, D(g)$, for all $g \in G$. We soon check that D' is an ω'-representation, where

$$\omega'(g_1, g_2) = \frac{\alpha(g_1)\alpha(g_2)}{\alpha(g_1 g_2)}\, \omega(g_1, g_2), \tag{3}$$

for all $g_1, g_2 \in G$. We say that ω' is equivalent to ω with gauge function α. Note that α is not unique in that we can multiply it by any unitary linear character of G without changing ω'. Within each equivalence class of multipliers there are some which satisfy $\omega(e, g) = \omega(g, e) = 1$, for all $g_1, g_2 \in G$. Henceforth we shall only deal with such normalized multipliers.

If we express a multiplier in the form $\omega(g_1, g_2) = \exp[2\pi i \phi(g_1, g_2)]$, where $\phi(g_1, g_2) \in [0, 1)$, for $g_1, g_2 \in G$, then the condition (2) and the normality condition become

$$\phi(g_1, g_2) + \phi(g_1 g_2, g_3) \equiv \phi(g_1, g_2 g_3) + \phi(g_2, g_3), \bmod 2, \tag{4}$$

for all $g_1, g_2, g_3 \in G$, and

$$\phi(e, g) = \phi(g, e) = 0, \tag{5}$$

for all $g \in G$.

Using pointwise multiplication of multipliers the equivalence classes form an Abelian group called the Schur Multiplicator $M(G)$. In this group the identity element is the class consisting of the so-called trivial multipliers, which are of the form

$$\omega(g_1, g_2) = \frac{\alpha(g_1 g_2)}{\alpha(g_1)\alpha(g_2)} , \tag{6}$$

for some gauge function α and for all $g_1, g_2 \in G$. Note that two multipliers are equivalent if their quotient is trivial. Expressing the gauge function in the form $\alpha(g) = \exp[2\pi i\theta(g)]$, $g \in G$, the condition for triviality becomes

$$\phi(g_1, g_2) \equiv \theta(g_1 g_2) - \theta(g_1) - \theta(g_2), \text{ mod } Z, \tag{7}$$

for all $g_1, g_2 \in G$.

There are numerous applications of projective representations, some of which are indicated in [3] , as well as a pure theory. However, the main motivation for the present paper is connected with the construction of irreducible representations of crystallographic space groups. In the theory, as expounded by Bradley and Cracknell [7] and summarized by the present author elsewhere in these proceedings, the computation of projective representations of certain point groups for given multipliers is required. Without any special simplifying features this can be quite an onerous task, and certainly will be if increasing use is to be made of the 4-dimensional space groups - recall that the largest 4-dimensional crystallographic point group has order 1152. However, as years go by, an increasing number of multipliers and associated projective representations of various groups are being computed - for a recent survey of results in this field see [5]. Therefore the following situation can arise: We wish to find the irreducible ω-representations of a given group G for a known multiplier ω. Suppose we already know a representative or standard set of multipliers and their associated irreducible projective representations. ω must be equivalent to one such standard multiplier via some gauge function. It follows that we can write down the irreducible ω-representations of G provided we can identify to which standard multiplier ω is equivalent and also that we can compute a gauge function. Equivalently we can pose the question: When is a function ϕ satisfying (4) and (5) representable in the form (7), and if so how can we determine the gauge function θ? The answer to these

questions, where G is a finite solvable group, is a special case of the solution to a problem discussed in the author's doctoral thesis (Oxford, 1971), but previously unpublished.

There are in fact other applications of the more general separation problem, namely: 1) The determination of non-primitive translations for non-symmorphic space groups: 2) The determination of the multipliers of symmorphic space groups, see Backhouse [2] and Bradley [6] . It therefore seems worthwhile making a statement of the general problem and its solution - this we do in § 2 . We have not included proof of our main result, for it is in fact only a little more involved than the proof given in [2] . Finally in § 3 we specialize to the trivial multiplier problem explained above.

2. The general separation problem for solvable groups

Let P be a finite group. An Abelian group A, written additively, is said to be a P-group if for each pair $(p, a) \in P \times A$ there exists an element $pa \in A$, which satisfies

(i) $ea = a$, for all $a \in A$, where e is the identity in P;
(ii) $p_1(p_2a) = (p_1p_2)a$, for all $p_1, p_2 \in P$, $a \in A$;
(iii) $p(a + b) = pa + pb$, for all $p \in P$, $a, b \in A$.

We have some further definitions

(a) A mapping $u: P \to A$ is a 1-cocycle if

$$u(p_1p_2) = u(p_1) + p_1u(p_2) , \tag{8}$$

for all $p_1, p_2 \in P$, and where $u(e) = 0$.
(b) A 1-cocycle u is a 1-coboundary if there exists a fixed element $a \in A$, is a 0-cochain, such that

$$u(p) = a - pa, \tag{9}$$

for all $p \in P$.
(c) A mapping $v : P \times P \to A$ is a 2-cocycle if

$$v(p_1, p_2) + v(p_1p_2, p_3) = v(p_1, p_2p_3) + p_1v(p_2, p_3), \tag{10}$$

for all $p_1, p_2, p_3 \in P$, and where $v(p, e) = v(e, p) = 0$ for all $p \in P$.
(d) A 2-cocycle v is a 2-coboundary if there exists a map $u: P \to A$, a 1-cochain, such that

$$v(p_1, p_2) = u(p_1 p_2) - u(p_1) - p_1 u(p_2), \qquad (11)$$

for all $p_1, p_2 \in P$.

In these terms a set of non-primitive translations for a non-symmorphic space group is a 1-cocycle where P is the point group, A is the additive group of all real translation vectors modulo the discrete lattice translations, and the action of P on A is the one induced by the natural action of the point group on vectors. Note that if the origin is shifted then the set of non-primitive translations is altered by the addition of a 1-coboundary. Such considerations are developed more fully by Ascher and Janner [1] .

Multipliers also fit into this scheme. In fact (4) and (5) say that ϕ is a 2-cocycle where A is the additive group of reals modulo the integers Z and the action of P on A is trivial. Also (7) says that ϕ is trivial if it is a 2-coboundary and the associated 1-cochain gives the gauge function θ . It should be noted that the 1-cochain is only unique up to the addition of a 1-cocycle.

Thus the question posed in §1 is a special case of: How to recognize when a 2-cocycle is a 2-coboundary and if so how to determine an associated 1-cochain? This we can do when P is solvable.

We say that a finite group P is <u>solvable</u> if there is a subnormal series

$$\{ e \} = P_0 \lhd P_1 \lhd P_2 \ldots \lhd P_{r-1} \lhd P_r = P,$$

where P_i/P_{i-1} is a cyclic group of order n_i, $i = 1, 2, \ldots r$. All of the 32 three-dimensional crystallographic point groups satisfy this condition. However, in higher dimensions this is not always true. The simplest example is the alternating group A_5 , of order 60, which is the proper symmetry group of the regular pentahedroid in 4-dimensions - see Backhouse and Gard [4] .

If P is solvable it can be built up from a finite set of generators $\{ p_i, i = 1, 2, \ldots r \}$ of finite order. We mean this in the following sense: The elements of P_i can be expressed uniquely in the form pp_i^s, for $0 \leq s \leq n_i - 1$, where $p \in P_{i-1}$, $i = 1, 2, \ldots r$. We say that p_i generates P_i from P_{i-1}. Generators are not unique, but we assume that some definite choice has been made.

We can now state our algorithm for the <u>separation</u> of a 2-cocycle as the 2-coboundary of a 1-cochain, where this is possible.

Algorithm

Let v be a 2-cocycle. We can decide whether or not v is a 2-coboundary by employing the following construction for the separations of v:

(i) Choose $u(e) = 0$.

(ii) Assume all separations of v on P_{i-1} are known. For each such separation find all possible values of $u(p_i)$ satisfying

$$u(p_i^{n_i}) = [\, v(p_i^{n_i-1}, p_i) + \dots + v(p_i, p_i) \,] + [\, p_i^{n_i-1} + \dots + p_i + e \,]\, u(p_i), \tag{A1}$$

where $u(p_i^{n_i})$ is known because $p_i^{n_i} \in P_{i-1}$.

(iii) From the possible values of $u(p_i)$ satisfying (A1) select those which are compatible with the corresponding separation for v on P_{i-1} by insisting that

$$v(p_j, p_i) + u(p_j) + p_j u(p_i) = v(p_i, p_i^{-1} p_j p_i) + u(p_i) + p_i u(p_i^{-1} p_j p_i) , \tag{A2}$$

for $j = 1, 2, \dots i - 1$.

(iv) For each separation on P_{i-1} which successfully passes the test (A2) and for each of the corresponding values $u(p_i)$, extend the separation to P_i by defining

$$u(pp_i^s) = v(p, p_i^s) + u(p) + pu(p_i^s) , \tag{A3}$$

where $p \in P_{i-1}$ and

$$u(p_i^s) = [v(p_i^{s-1}, p_i) + \dots + v(p_i, p_i)] + [p_i^{s-1} + \dots + p_i + e\,]\, u(p_i), \tag{A4}$$

for $0 \leq s \leq n_i$.

(v) The index i is allowed to run from 1 to r, except that, if for some value we can find no solutions to both (A1) and (A2), then we will have shown that v is not a 2-coboundary. Otherwise we will have found all separations of v on P.

Now let us look at the specialization of this algorithm to the multiplier problem.

3. The multiplier algorithm

We recall that for a multiplier the action of P on A is trivial. An important consequence of this is that $u(p_i)$ cancels from condition (A2), implying that this compatibility relation can be tested before $u(p_i)$ is constructed.

Let ϕ be a multiplier for P. We can decide whether or not ϕ is trivial by carrying out the following construction:

(i) Choose $\theta(e) = 0$

(ii) From the known separations of ϕ on P_{i-1} select those, if any, which satisfy

$$\phi(p_j, p_i) + \theta(p_j) \equiv \phi(p_i, p_i^{-1} p_j p_i) + \theta(p_i^{-1} p_j p_i) \bmod Z \qquad \text{(B1)}$$

for $j = 1, 2, \ldots i - 1$.

(iii) For each separation of ϕ on P_{i-1} which satisfies (B1), $\theta(p_i)$ can take any of the n_i values

$$[\theta(p_i^{n_i}) - (p_i^{n_i - 1}, p_i) \ldots - \phi(p_i, p_i) + k\,]/n_i \bmod Z, \qquad \text{(B2)}$$

for $k = 0, 1, \ldots n_i - 1$.

Now extend θ to the whole of P_i by the formulae

$$\theta(pp_i^s) \equiv \phi(p, p_i^s) + \theta(p) + \theta(p_i^s) \bmod Z, \qquad \text{(B3)}$$

where $p \in P_{i-1}$, and

$$\theta(p_i^s) \equiv \phi(p_i^{s-1}, p_i) + \ldots + \phi(p_i, p_i) + s\,\theta(p_i) \bmod Z, \qquad \text{(B4)}$$

for $0 \leqslant s \leqslant n_i$.

Iteration of this determines whether or not ϕ is trivial, and if so, produces all gauge functions θ. Some comments on the practicalities of the algorithm are worth making. For example, in certain circumstances, a modification to the way in which we apply (B1) can lead to saving of effort. As set out above we think of i as being temporarily fixed and then we apply (B1) for $j = 1, 2, \ldots i - 1$. On the other hand, for a given j, there may be values of $i > j + 1$ for which $p_i^{-1} p_j p_i \in P_j$. In such cases (B1) can be applied before we get to the ith stage and certain separations eliminated earlier than they would be if we applied the algorithm as it stands. A more significant economy of effort, however, derives from the observation that two separations of

the same multiplier can differ only by a linear character. To take advantage of this we should prepare the ground by computing the linear characters of all groups in the subnormal series for P. At the ith stage of the algorithm we do not have to test all separations of ϕ on P_{i-1}. In fact if a particular separation fails the test (B1), then so will all separations which differ from it by a linear character of P_{i-1}. On the other hand, once we have found a separation which passes the test (B1), we can immediately write down all others by the addition of linear characters.

Finally we remark that our algorithm could be adapted for use on an electronic computer.

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