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Plenary Lecture

A NEW EXAMPLE OF AN UNSTABLE SYSTEM BEING STABILIZED BY RANDOM PARAMETER NOISE

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Summary

It is shown that any two-dimensional linear system having real eigenvalues a and b satisfying b 0 a and a + b 0 can be made exponentially stable by applying one single real noise source.

The destabilizing impact of random noise in linear and nonlinear technical and physical systems is well-known (see, e.g., Hasminski [3], Kozin [5], Arnold and Wihstutz [1]).

However, for chemical and biological systems it is often argued that parameter noise plays a positive role for the stability behaviour of the system. But examples supporting this argument are still quite rare. Hasminski ([3], p.280) constructed a two-dimensional unstable system which can be stabilized by two independent white noise sources.

In this note, we present a new class of unstable deterministic systems in the plane which can be stabilized by applying just one single real (i.e. non-white) noise source.

Since under general conditions the stability properties of a nonlinear system are the same as those of the corresponding linearized system (Hahn [2], p.319 ff), we restrict ourselves to linear systems.

Furthermore, a stabilizable system must have dimension $n \geqq 2$, since for $n = 1$

$$\dot{x}_t = (a+u_t)x_t , \quad a \text{ real number},$$

yields

$$x_t = x_o \exp t(a + \frac{1}{t} \int_o^t u_s ds) .$$

For any zero mean stationary ergodic random process u_t we have

$$\lim_{t\to\infty} \frac{1}{t} \int_o^t u_s ds = 0 \quad \text{with probability 1,}$$

so that the deterministic stability behavior is being conserved.

<u>Theorem</u>. Given any two-dimensional deterministic <u>unstable</u> linear autonomous system $\dot{z}=Az$ with two real eigenvalues a,b satisfying $b<0<a$ and $a+b<0$. Then one can choose a real noise u_t (i.e. a zero mean stationary ergodic random process) and a matrix B so that the trivial solution of the parameter-excited system

$$\dot{z} = (A + u_t B)z$$

is exponentially <u>stable</u> with probability 1. More precisely, there exists a constant R satisfying $(a+b)/2 < R < 0$ such that for any solution z_t

$$\lim_{t\to\infty} \frac{1}{t} \log|z_t| = R \quad \text{with probability 1.}$$

<u>Proof</u>.

1. Without restricting the generality we assume for the deterministic system the uncoupled form

$$\dot{z} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} z, \quad b<0<a, \quad a+b<0, \qquad (1)$$

which can always be derived by coordinate transformation, while for the perturbed system we take

$$\dot{z} = \left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} + u_t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) z = \begin{pmatrix} a & u_t \\ -u_t & b \end{pmatrix} z. \qquad (2)$$

2. As real noise in (2) we choose the well-known Ornstein - Uhlenbeck process (so-called coloured noise), i.e. a stationary Gaussian Markov process u_t with spectral density

$$f(\lambda) = \frac{\sigma^2}{2\pi} \frac{1}{\lambda^2+\alpha^2}, \qquad \sigma^2, \alpha > 0,$$

and the Gaussian distribution with mean zero and variance $D^2 = \sigma^2/2\alpha$ at any time t.

3. We use the method developed by Wihstutz [6] (see also Arnold and Wihstutz [1]) for dealing with the stability of (2). After introducing polar coordinates $x = r\cos\varphi$, $y = r\sin\varphi$, we obtain after simple calculations

$$|z_t| = |z_o| \exp \int_o^t Q(\varphi_s)ds, \tag{3}$$

where

$$Q(\varphi) = (b-a)\sin^2\varphi + a,$$

while the angle φ satisfies the nonlinear differential equation

$$\dot{\varphi}_t = \frac{b-a}{2} \sin 2\varphi_t - u_t. \tag{4}$$

If we had for a certain value of D^2

$$\lim_{t\to\infty} \frac{1}{t} \int_o^t Q(\varphi_s)ds = R < o,$$

the proof would be finished .

4. The main theoretical difficulty is that - unlike in the case of white disturbances - the solution z_t of (2) is not a Markov process anymore. But (u_t, φ_t) is a Markov process with state space $\mathbb{R} \times [o,2\pi)$, where the lines $\varphi = o$ and $\varphi = 2\pi$ are identified. The pair (u_t, φ_t) is a degenerate two-dimensional diffusion prodess since there is no diffusion component in φ-direction. At the so-called 'switching curves'

$$\frac{b-a}{2} \sin\varphi = u$$

$\dot{\varphi}$ changes sign (see Figure 1).

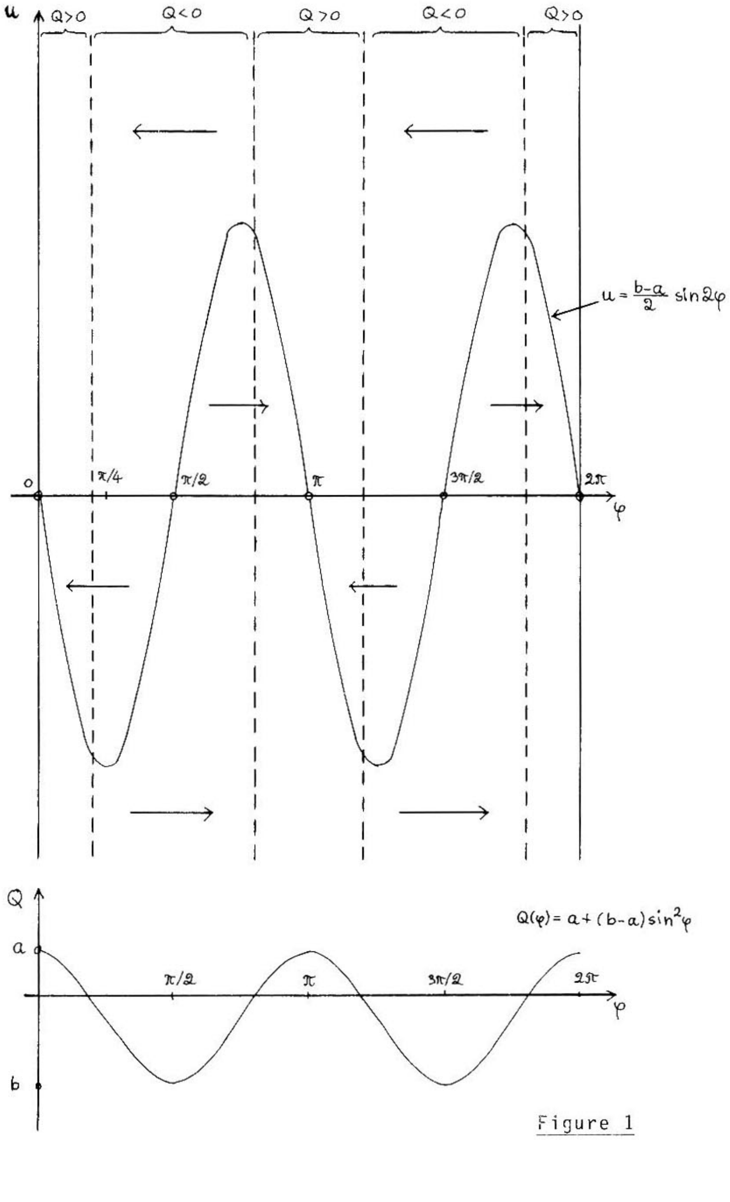

u
Q>0
Q<0
Q>0
Q<0
Q>0
u = (b-a)/2 sin2φ
0
π/4
π/2
π
3π/2
2π
φ
Q
a
b
Q(φ) = a + (b-a) sin²φ
π/2
π
3π/2
2π
φ

Figure 1

While (u_t, φ_t) is travelling around in the state space, φ_t picks-up 'mass' $\int_0^t Q(\varphi_s)ds$ which completely determines the growth of z_t. We will choose D^2 so that (u_t, φ_t) is pushed into areas where Q is, on the average, negative.

5. By a theorem of Kliemann [4] there exists a unique solution φ_t^o of (4) depending only on the noise u_s for $s \leqq t$ such that (u_t, φ_t^o) is a stationary ergodic Markov process. If μ is the distribution of φ_t^o on $[o, 2\pi)$, there is a law of large numbers saying that for any solution φ_t of (4)

$$\lim_{t\to\infty} \frac{1}{t} \int_o^t Q(\varphi_s)ds = \int_o^{2\pi} Q(\varphi)\mu(d\varphi) = R$$

with probability 1. Thus (3) yields

$$\lim_{t\to\infty} \frac{1}{t} \log |z_t| = R.$$

We are now investigating the sign of the growth rate R.

6. For 'small' noise, i.e. D^2 small, so that $|u_t| < \frac{a-b}{2}$ with probability close to 1, (u_t, φ_t) will stay around the attracting branches of the switching curve $u = \frac{b-a}{2}\sin 2\varphi$. Since everything is π-periodic, we need to consider only the branch through (o,o). A first approximation of φ_t is therefore

$$\psi_t = \frac{1}{2} \text{arc sin} \frac{2u_t}{b-a} .$$

Using this for the calculation of R we obtain

$$R \approx \frac{a+b}{2} + \frac{1}{\sqrt{2\pi D^2}} \int_{|u| \leqq (a-b)/2} \sqrt{\left(\frac{a-b}{2}\right)^2 - u^2} \exp\left(-\frac{u^2}{2D^2}\right) du$$

$$\approx a - \varepsilon \frac{a-b}{2} > o$$

provided ε is small enough which can always be accomplished by taking D^2 small enough. Therefore, for small noise the system (2) is still unstable.

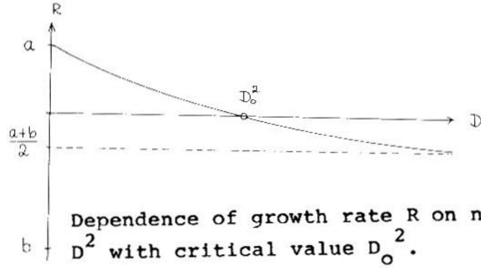
7. Now D^2 is being increased to a level for which u_t spends a proportion of time close to 1 outside a big interval $|u| \leqq C$. In other words, for the average behavior of (u_t, φ_t) it does not really matter what happens around the switching curves. Since u_t is most of the time very large, we have

$$\dot{\varphi}_t \approx - u_t,$$

i.e. φ_t is rapidly circling around, changing directions if u_t changes sign. This means that the distribution of φ_t is approximately uniform in $[o, 2\pi)$. This entails

$$R \approx \frac{1}{2\pi} \int_o^{2\pi} Q(\varphi) d\varphi \quad = \quad \frac{a+b}{2} < o \quad .$$

We can conclude that there must be a critical noise variance D_o^2 depending on a and b with $R(D_o^2) = o$, so that we have the situation shown in Figure 2.



Dependence of growth rate R on noise variance D^2 with critical value D_o^2.

FIGURE 2

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