match no. 6 pp. 117 - 135 1979

SOME PROPERTIES OF SEMI-REGULAR GRAPHS

C.W. Evans

Department of Mathematics
Portsmouth Polytechnic
England

Semi-regular graphs habe been applied to chemistry by Balaban (1) and Balaban and Kerek (3) in the form of combination graphs and convolution graphs respectively. Here the notion of a semi-regular graph is formalised and necessary conditions are found for the existence of such graphs with prescribed intersection matrices. If such graphs are to be finite then a further necessary condition must be satisfied. This condition is investigated fully in the case where the graph is to contain vertices with three different degrees.
An operation on a graph which preserves degree (but not necessarila semi-regularity) is considered and necessary and sufficient conditions for one graph to be obtainable from another by a sequence of these operations are used to alluminate the structure of semi-regular graphs. Infinite formation are considered briefly and the discussion is conclused with an indication of some of the more general properties of semi-regular graphs.

§1. Finite Formations

Unless otherwise stated, all graph will be supposed connected and finite. Multiple edges will be permitted but loops will be excluded.

Suppose Γ, υ and V denote a graph, its set of vertices and the number of vertices respectively. Suppose also that Γ contains vertices of degree $d_1, \ldots\ldots d_r$ where

$$d_1 < d_2 < \ldots\ldots < d_r. \qquad (1)$$

Equalities can be included if it is required that some vertices of the same degree should be distinguished from one another. If β_i denotes the number of vertices of degree d_i $(i \in \{1,\ldots,r\})$ in υ then

$$V = \sum_{i=1}^{r} \beta_i. \qquad (2)$$

Definition 1

A graph Γ is called semi-regular if and only if given any vertex v υ of degree d_i $(i \in \{1,\ldots,r\})$ the number of vertices of degree d_j which are adjacent to v is a constant α_{ij} independent of the choice of v.

This definition is also suitable for infinite graphs of finite degree.

Example 1

Grötsch's graph |5| satisfies the conditions for a semi-regular graph.

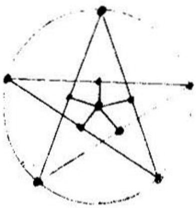


Fig. 1

Here $d_1 = 3$, $d_2 = 4$, $d_3 = 5$
and $\beta_1 = 5$, $\beta_2 = 5$, $\beta_3 = 1$.
So $\alpha_{11} = 0, \alpha_{12} = 2, \alpha_{13} = 1$
$\alpha_{21} = \text{"}, \alpha_{22} = 2, \alpha_{23} = 0$
$\alpha_{31} = 5, \alpha_{32} = 0, \alpha_{33} = 0.$

Definition 2

Suppose Γ is a semi-regular graph then its intersection matrix M is defined by $M = |\alpha_{ij}|$ where $i,j \in \{1,.r\}$.

Hence Grötsch's graph has intersection matrix,

$$\begin{bmatrix} 0 & 2 & 1 \\ 2 & 2 & 0 \\ 5 & 0 & 0 \end{bmatrix}$$

It should be noted that all graphs are in a sense semi-regular, for in the case of a regular graph of degree h the intersection matrix $M = [h]$.

Whereas in the case of a highly irregular graph the vertices may all be regarded as degree distinct and so if $V = n$ then $d_1 \leqslant d_2 \leqslant \ldots\ldots < d_n$ and M is of order n x n.

In neither of these extreme cases does the semi-regular concept give further insight into the structure of the graph. However to require $d_i \neq d_j$ unless $i = j$ would be too restrictive for then it would prohibit discussion if many graphs which would be nearly semi-regular.

Lemma 1

Suppose that Γ is a semi-regular graph with intersection matrix M then (in the previous notation)

(a) the row sums give the degrees. That is,

$$\sum_{j=1}^{r} \alpha_{ij} = d_i. \tag{3}$$

(b) $\beta_j > \alpha_{jj}$ and if Γ has no multiple edges then

$$\beta_j \geqslant \max \{ \alpha_{ij} : i \in \{ 1, \ldots, r\}\}. \tag{4}$$

(c) suppose $|\delta_{ij}|$ is defined by $\delta_{ij} = 0$ if $\alpha_{ij} = 0$ and $\delta_{ij} = 1$ if $\alpha_{ij} \neq 0$ then $|\delta_{ij}|$ is symmetric.

Proof The properties follow immediately from the definitions. #

Suppose Γ is to be finite then in order to decide which intersection matrices are feasible the following formula will be used. This formula is easily derived by counting the vertices in Γ.

$$V = \beta_i \sum_{\alpha_{ij} \neq 0} \frac{\alpha_{ij}}{\alpha_{ji}} + \sum_{\alpha_{ij} \neq 0} \beta_j \qquad (5)$$

for every $i \in \{1,....,r\}$.
From (2) and (5) it follows that

$$\beta_i \sum_{\alpha_{ij} \neq 0} \frac{\alpha_{ij}}{\alpha_{ji}} - \sum_{\alpha_{ij} \neq 0} \beta_j = 0 \qquad (6)$$

for every $i \in \{1,....,r\}$.

These equations represent a general condition. If a finite semi-regular graph Γ is to exist then the equation must have a solution $\beta_1,....,\beta_r$ (not all zero). This condition will be called the feasibility condition. The admissible values for β_i ($i \in \{1,...,r\}$) will be called degree numbers and can be written as an ordered r-tuple $(\beta_1,...,\beta_r)$.

If $\beta_1,...,\beta_r$ have no common factor then these degree numbers will be called formation numbers. If a graph Γ exists with a given set of formation numbers then it will be called a formation. In any given situation a formation might be a molecular diagram, a crystal, a diagram representing the interconversions of chemical species |3|. A formation may have loops and multiple edges and so condition (4) must be used to produce suitable degree numbers. Suppose p is the smallest positive integer such that Γ has no multiple edges and the degree numbers are $(p\beta_1,...,p\beta_r)$ where $(\beta_1,...,\beta_r)$ are the formation numbers. Then p will be cakked the formation factor and Γ a <u>strict</u> formation.

§ 2 A special case

The implications of the feasibility condition will now be considered in the special case r = 3 and $\alpha_{ij} \neq 0$ if $i \neq j$.

Denote $\frac{\alpha_{ij}}{\alpha_{ji}}$ by a_{ij} when $\alpha_{ij} \neq 0$ so that $a_{ji} = a_{ij}^{-1}$.

The equation (6) become:

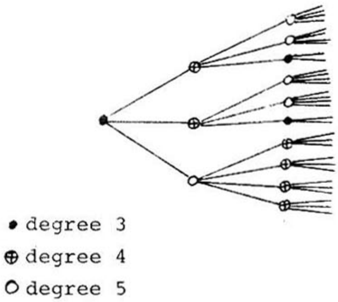
$$\begin{bmatrix} a_{12} + a_{13} & -1 & -1 \\ -1 & a_{12}^{-1} + a_{23} & -1 \\ -1 & -1 & a_{13}^{-1} + a_{23}^{-1} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For a non-zero solution the matrix of coefficients must be singular and it is a matter of direct verification to show that this implies $a_{13} = a_{12}a_{23}$ from which is obtained the condition $\alpha_{12}\alpha_{23}\alpha_{31} = \alpha_{32}\alpha_{21}\alpha_{13}$.

It is now possible to assert that there are no finite semi-regular graphs with certain intersection matrices. For example, consider the case $d_1 = 3$, $d_2 = 4$, $d_3 = 5$ and

$$M = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 4 & 0 \end{bmatrix}$$

The condition $\alpha_{12}\alpha_{23}\alpha_{31} = \alpha_{32}\alpha_{21}\alpha_{13}$ is not satisfied and so no finite formation exists. Clearly an infinite formation must exist for it is possible to construct a tree with this structure:



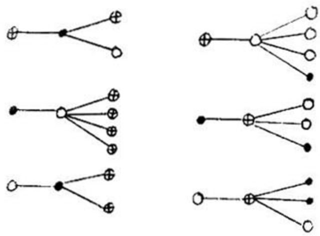


Fig. 2 Partial Tree Tree continuation diagramms

The existence of finite formation will now be considered in the case $d_1 = 3$, $d_2 = 4$, $d_3 = 5$ and the stronger condition $\alpha_{ij} = 0$ if and only if $i = j$. This means that no two vertices of the same degree are adjacent. The feasibility condition $\alpha_{12}\alpha_{23}\alpha_{31} = \alpha_{32}\alpha_{21}\alpha_{13}$ Together with (3) produces just two possible intersection matrices:

$$(a) \quad \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 0 \end{bmatrix} \qquad (b) \quad \begin{bmatrix} 0 & 2 & 1 \\ 3 & 0 & 1 \\ 3 & 2 & 0 \end{bmatrix}$$

The corresponding formation numbers are (1,1,1) and (3,2,1).

(a) The formation factor is 3 and the appropriate formations are:

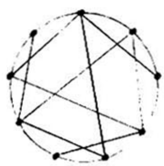


Fig. 3 Formation (1,1,1) Strict Formation (3,3,3)

This strict formation is unique. To see this, excise the three vertices of degree 5 and their incident edges. This operation reduces the degree of all the remaining vertices to 1. The residual graph therefore consists of three edges and the reconstruction is then forced.

(b) The formation numbers and the strict formation numbers coincide, further the strict formation is uniquely determinde. For if the vertex of degree 5 is excised together with its incident edges, the remaining graph has 3 vertices of degree 2 and 2 of degree 3. Only one of the two possible graphs produces a strict formation and this reconstruction is unique.

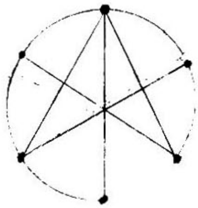


Fig. 4 Strict Formation (3,2,1)

Remark In general, formations are not uniquely determined by the formation numbers even when the formation is strict. For example, Grötzsch's graph and the planar graph shown in figure 5 are both formations with the same intersection matrix.

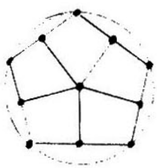


Fig. 5 A planar graph with the same intersection matrix as Grötzsch's graph.

In the special case r = 3, the original condition

C1: $\alpha_{ij} \neq 0$ if $i \neq j$

was strengthened by the further requirement

C2 $\alpha_{ij} = 0$ if $i = j$.

When in the case $d_1 = 3$, $d_2 = 4$. $d_3 = 5$, C1 alone is imposed a further 16 feasible matrices are obtained. These are listed in Table 1. On the other hand, when C2 alone is imposed a further 8 feasible matrices are obtained. These are listed in Table 2.

If neither the condition C1 nor the condition C2 is imposed then there are approximately a further 260 feasible matrices.

The case $d_1 = 1$, $d_2 = 2$, $d_3 = 3$ may be of particular interest in chemistry when graphs are applied to the determination of some molecular structures. Table 3 lists the 9 matrices which are feasible, together with the appropriate formations.

§ 3 Edge related graphs

An operation in a graph will now be described. This operation, which will be called edge adaptation, preserves the degree of each vertex and yet may produce multiple edges or loops and may even disconnect the graph.

Definition 2

(a) An edge adaptation of a graph Γ consists of deleting two edges $\{u_1,v_1\}$, $\{u_2,v_2\}$ and replacing them by the edges $\{u_1,v_2\}$ and $\{u_2,v_1\}$.

(b) Two graphs Γ_1, Γ_2 will be said to be edge related if and only if a graph isomophic to Γ_2 can be obtained from Γ_1 by applying a sequence of edge adaptations.

(c) Two graphs Γ_1 and Γ_2 will be said to be compatible if and only if there is a bijection ϕ: $\upsilon_1 \rightarrow \upsilon_2$ such that degree v = degree ϕ(v) for all v υ_1.

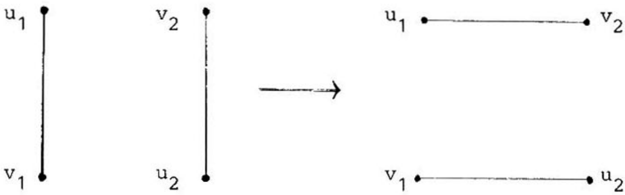


Fig. 6 Edge Adaptation

A necessary and sufficient condition for two graphs to be edge related is now obtained.

Theorem 1 Two graphs Γ_1 and Γ_2 are edge related if and only if they are compatible.

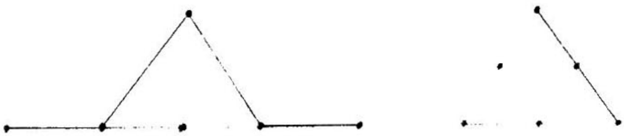
Proof Edge adaptation preserves the degree of the vertices involved and so it follows immediately that edge related graphs re compatible.

Suppose now Γ_1 and Γ_2 are compatible. Construct a graph with coloured edges as follows. Take an isomorphic copy Γ_1 and colour its edges red, then augment the edge set by blue edges so that the graph obtained when the red edges are suppressed is isomorphic to Γ_2. For every vertex in Γ the number of blue edges and the number of red edges incident with it are equal.

Suppose there are some pairs of vertices which are joined by both a blue edge and a red edge, then replace these edges by a single orange edge. It suffices to prove that Γ is edge related to a graph in which all the edges are orange. This will be shown by performing edge adaptations solely on red edges. If all the edges are orange there is nothing to prove. If not then there exists some vertex u_1 which is incident with both a red edge and a blue edge. Suppose the termini of these edges are v_1 and v_2 respectively then $v_1 \neq v_2$.

Now v_2 itself must be incident with a red edge. Suppose the terminus of this edge is u_2. Delete the red edges $\{u_1,v_1\}$, $\{u_2,v_2\}$ and replace them by the red edges $\{u_1,v_2\}$ and $\{u_2,v_1\}$. The edge $\{u_1,v_2\}$ is now rendered orange. Provided there are red edges in the graph the process may be continued, since Γ is finite the process terminates.

The following examples illustrate theorem 1, in fact the construction employed in the proof can be used to exhibit how compatible graphs are edge related.



Graphs with the same numbers of vertices, edge and circuits which are not edge related.

Butane and Isobutane are edge related.
The arrows indicate the process.

Fig. 7

In fact there is a distance function defined on any set of compatible graphs. Define dist (Γ_1, Γ_2) to be the length of the shortest sequence of edge adaptations required to obtain a graph isomorphic to Γ_2 from Γ_1. So that dist $(\Gamma_1, \Gamma_2) = 0$ if and only if Γ_1 is isomorphic to Γ_2 and dist $(\Gamma_1, \Gamma_2) = 1$ if and only if Γ_1 and Γ_2 are simply edge related. This distance function gives to any set of compatible graphs the structure of a metric space.
The next theorem shows how any finite semi-regular graph relates to its formation (if one exists).

Theorem 2

Suppose corresponding to a feasible intersection matrix M, Γ is any finite semi-regular graph with degree numbers $(\lambda_1,\ldots,\lambda_r)$ and that Γ^o is a formation with degree numbers $(\beta_1,\ldots,\beta_r)$ then

(1) $\lambda_1 = p\beta_i$ for some p and $i \in \{1,\ldots,r\}$.

(2) Γ is edge related to $\bigcup_{r=1}^{p} \Gamma_r^{\ o}$ where $\Gamma_r^{\ o}$ is isomorphic to Γ^o for $r \in \{1,\ldots,p\}$.

Proof (1) follow immediately from the definition of a formation.

Γ and $\bigcup_{r=1}^{p} \Gamma_r^{\ o}$ are compatible and so by theorem 1 it follows that these graphs are edge related.

Example 2

In |2| Balaban discussed the isomers of the hydrocarbon C_5H_8. In particular there are 5 corresponding to the formula $(CH_2)_3(CH)_2$.

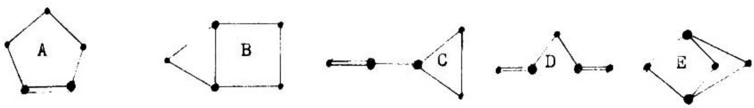


Fig. 8 Isomers of the form $(CH_2)_3(CH)_2$.

A general process will now be illustrated. Form a new graph with vertices A,B,C,D,E and define adjacency by $X \sim Y$ if and only if dist $(X,Y) = 1$. In the present case this produces the graph in Figure 9.

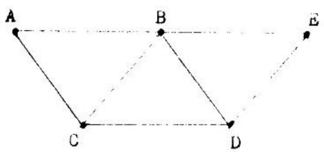


Fig. 9 The edge relationship between the isomers $(CH_2)_3(CH)_2$.

§ 4 Infinite Formations

In general provided the conditions of lemma 1 are satisfied then an infinite graph exists with the intersection matrix M. This is clear, for a tree can always be constructed with the intersection matrix. Those intersection matrices which do not correspond to a finite graph must therefore correspond to infinite graphs of some kind. These graphs need not be trees and in general there are many graphs with the dame intersection matrix. Consequently the topological restrictions are not so rigid and in any application other constraints (for example geometric) may favour one structure over another.

Example 3

The intersection matrix:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix}$$

does not correspond to a finite graph and so the corresponding structure must be infinite. Figure 10 shows a diagram which is not a tree in which the pattern for continuation is established.

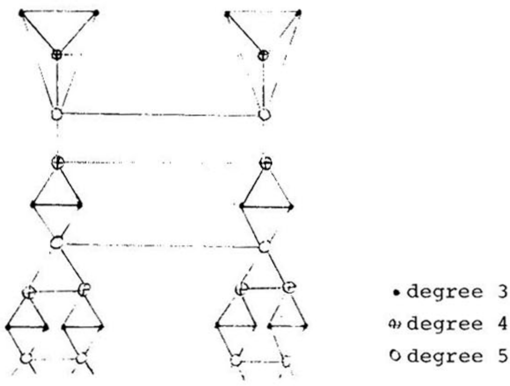


Fig. 10 An infinite formation which is not a tree.

Table 4 lists those intersection matrices, in the case $d_1 = 3$, $d_2 = 4$, $d_3 = 5$ which must correspond to infinite formations. Even if such formations cannot grow beyond a certain size the irregularity may be confined to the boundary.

Example 4

The intersection matrix:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

corresponds to an infinite formation.

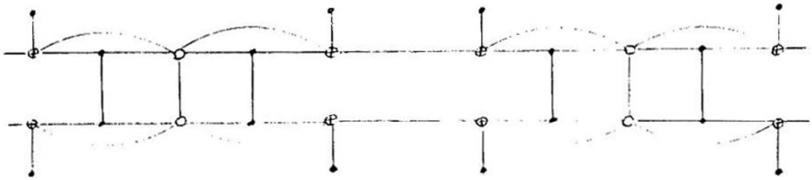


Fig. 11 An incomplete infinite formation. It can be regarded as a formation with restricted growth.

§ 5 Girth

Semi-regular graphs have not hitherto been investigated by graph theorists and yet it seems that many of the concepts introduced for regular graphs remain intersting when semi-regular graphs are considered. From the chemical point of view the existence of a topologically valid formation does not ensure a chemical counterpart, for further structural considerations arise. The concept of girth may therefore be of use, particularly if small circuits need to be avoided.

Given any feasible intersection matrix, it is possible to look for corresponding finite semi-regular graphs which have girth exceeding some number g. By theorem 2 the degree numbers must be divisible by the formation numbers.

A graph with the minimum number of vertices consistent with these conditions (if such a graph exists) is a generalisation of the idea of a cage (Tutte |6|). Such a graph will be called a semi-regular cage.

Suppose &(m,n), m n denotes the class of graphs with intersection matrix:

$$\begin{bmatrix} 0 & m \\ n & 0 \end{bmatrix}$$

A member of &(m,n) will be denoted by $\Gamma(m,n)$. All graphs in this class are bipartite and when n and m are relatively prime the formation numbers are (n,m). It follows that if $m \neq n$ no member of this family hamiltonian. Furthermore, if n and m are relatively prime, then by theorem 2 the number of vertices in any member of &(m,n) is divisible by m+n. When n = m the graphs become regular bipartite graphs.

Suppose that $\Gamma_g(m,n)$ is a semi-regular cage with girth g (necessarily even) and $V_g(m,n)$ vertices.

Conjecture:

$$V_g(m,m) \leq V_g(m,n) \leq V_g(n,n)$$

This conjecture can be verified directly for small values of g in the case of the family &(3,4).

(1) $\Gamma_4(3,4)$ is the graph $K_{3,4}$. So $V_4(3,3) = 7$. Further by |6| $\Gamma_4(3,3) = K_{3,3}$ and $\Gamma_4(4,4) = K_{4,4}$ si that $V_4(3,3) = 6$ and $V_4(4,4) = 8$.

(2) $\Gamma_6(3,4)$ is the graph shown in Figure 12 whereas, by 6 $V_6(3,3) = 14$ and $V_6(4,4) = 26$.

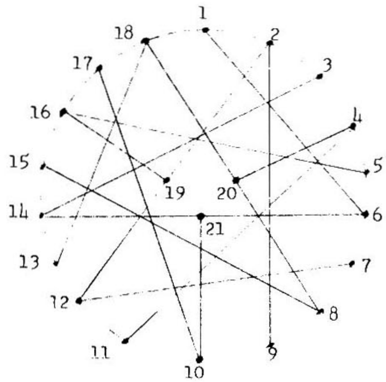


Fig. 12 $\Gamma_6(3,4)$

If in Figure 12 the vertices labelled 19, 20 and 21 are excised together with their incident edges, a regular trivalent graph is obtained which is known to geometers as the Levi graph of the Pappus configuration 4 .

§ 6 Possible Chemical Implications

Purely topological questions of the kind that have been discussed raise chemical questions. For instance an application of semi-regular graphs would arise if there is a chemical mechanism whereby certain molecules always bond in a definite pattern with other molecules. Certainly sone compounds exhibit semi-regular structure (e.g. methane, ethane, methyl boronic acid, diketopiperazine, diborane, tetrameric thallium ethoxide and many others). If on the other hand the mechanism is weak, so that only a propensity to bond in a semi-regular fashion holds, then incomplete formations might be produced.

Possibly a chemical mechanism equivalent to edge adaptation exists. If this is the case then it is now clear that some isomers cannot be produced from a given isomer using this mechanism without first forming other isomers. For example in the case of $(CH_2)_3(CH)_2$, it has been shown that A cannot be converted to E without first either existing as B or becoming C and then D.

TABLE 1 A complete list of feasible intersection matrices in the case $d_1=3$, $d_2=4$, $d_3=5$ for which $\alpha_{ij} \neq 0$ if $i \neq j$.

Intersection matrix	Formation numbers	Formation factor
$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 0 \end{bmatrix}$	(1,1,1)	3
$\begin{bmatrix} 0 & 2 & 1 \\ 3 & 0 & 1 \\ 3 & 2 & 0 \end{bmatrix}$	(3,2,1)	1
$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}$	(2,1,1)	4
$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix}$	(1,1,1)	4
$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 2 & 1 \end{bmatrix}$	(2,2,1)	2
$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}$	(1,1,1)	4
$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$	(1,1,1)	2
$\begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix}$	(2,2,1)	2
$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix}$	(1,1,2)	2
$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$	(1,1,1)	3
$\begin{bmatrix} 0 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix}$	(1,2,1)	3
$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 0 & 1 \\ 3 & 1 & 1 \end{bmatrix}$	(3,1,1)	2
$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 2 & 2 & 1 \end{bmatrix}$	(2,1,1)	2
$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 3 & 1 \end{bmatrix}$	(1,1,1)	4
$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 1 & 1 & 3 \end{bmatrix}$	(2,1,2)	2
$\begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 2 \\ 2 & 1 & 2 \end{bmatrix}$	(2,1,2)	2
$\begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 2 \end{bmatrix}$	(1,1,1)	3
$\begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix}$	(1,1,1)	4

TABLE 2 A complete list of feasible intersection matrices in the case $d_1=3$, $d_2=4$, $d_3=5$ for which $\alpha_{ij} = 0$ if $i = j$ (excluding those included in Table 1).

Intersection matrix	Formation numbers	Formation factor
$\begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 4 \\ 1 & 4 & 0 \end{bmatrix}$	(1,3,3)	2
$\begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 4 \\ 2 & 3 & 0 \end{bmatrix}$	(8,9,12)	1
$\begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 4 \\ 3 & 2 & 0 \end{bmatrix}$	(2,1,2)	2
$\begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 4 \\ 4 & 1 & 0 \end{bmatrix}$	(16,3,12)	1
$\begin{bmatrix} 0 & 2 & 1 \\ 4 & 0 & 0 \\ 5 & 0 & 0 \end{bmatrix}$	(10,5,2)	1
$\begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 3 \\ 0 & 5 & 0 \end{bmatrix}$	(5,15,9)	1
$\begin{bmatrix} 0 & 3 & 0 \\ 2 & 0 & 2 \\ 0 & 5 & 0 \end{bmatrix}$	(10,15,6)	1
$\begin{bmatrix} 0 & 3 & 0 \\ 3 & 0 & 1 \\ 0 & 5 & 0 \end{bmatrix}$	(5,5,1)	1
$\begin{bmatrix} 0 & 1 & 2 \\ 4 & 0 & 0 \\ 5 & 0 & 0 \end{bmatrix}$	(20,5,8)	1

TABLE 3 A complete list of feasible intersection matrices in the case $d_1=1$, $d_2=2$, $d_3=3$.

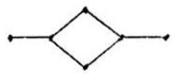
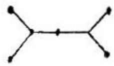
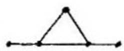
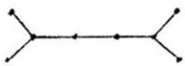
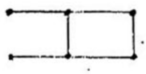
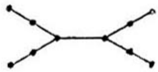
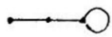
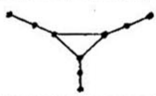
Intersection matrix	Formation numbers	Formation factor	Formation	Strict formation
$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$	(1,1,1)	2		
$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 2 & 1 & 0 \end{bmatrix}$	(4,1,2)	1		formation
$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$	(2,1,2)	1		formation
$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$	(1,2,1)	1		formation
$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$	(2,1,1)	2	none	
$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$	(1,1,1)	2	none	
$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$	(2,2,1)	2	none	
$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$	(1,1,1)	3		
$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 3 & 0 \end{bmatrix}$	(3,3,1)	1		formation

TABLE 4 A complete list of intersection matrices which must correspond to infinite formations in the case $d_1 = 3$, $d_2 = 4$, $d_3 = 5$. An ordered triple consisting of the elements $(\alpha_{11}, \alpha_{22}, \alpha_{33})$ on the leading diagonal is listed. The remaining entries in the intersection matrix must satisfy:

(1) $\alpha_{ij} \in \{1,2,3,4,5\}$.

(2) $\sum_{j=1}^{3} \alpha_{ij} = d_i$ for $i \in \{1,2,3\}$.

(1,1,1)	(1,1,3)	(1,2,2)	(0,2,1)	(0,1,2)	(0,2,3)
(1,0,2)	(1,1,0)	(1,2,0)	(0,0,1)	(0,0,3)	(0,2,0)
(1,0,0)					

References

|1| Balaban, A.T. Combinatorial Patterns. Rev. Roum. Math. Pures et Appl. 17, 1, p. 3-15 (1972)

|2| Balaban, A.T. Graphs of degree four or less, isomers of annulenes and nomenclature of bridged polycyclic structures. Rev. Roum. de Chemie 18, 4, p. 635-653 (1973).

|3| Balaban, A.T. and Kerek, F. Graphs of parallel and/or substitution reactions. Rev. Roum. de Chemie 19, 4, p. 631-647 (1974).

|4| Coxter, H.S.M. Self dual configurations and regular graphs. Bull. Amer. Math. Soc. 56, p. 413-455 (1950).

|5| Grötzsch, H Ein Dreifarbensatz für dreikreis freie Netze auf der Kugel. Wis. Z. Martin Luther Univ. Halle-Wittenberg. Math. Naturwiss. 8, p. 109-120 (1958).

|6| Tutte, W.T. Connectivity in Graphs. University of Toronto Press, Mathematical expositions no. 15 (1966).