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ON GRAPHS AND THEIR ENUMERATION III

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Summary: Continuing the work of part I and part II it is shown, how the representation theory of generalized wreath products can serve for a generalization and a unified treatment of the enumeration theories of Redfield, Pólya and de Bruijn.

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1. Introduction

We recall from part II that we considered finite sets X and Y, and correspondingly the set Y^X of all the mappings ϕ from X into Y:

$$Y^X := \{ \phi \mid \phi: X \to Y\}.$$

We denoted by S_X and S_Y the symmetric groups on X and Y.

If G and H denote permutation groups on Y and X, i.e. $G \leq S_Y$, $H \leq S_X$, then they induce various permutation groups on Y^X, and the problem is to enumerate their orbits or even give a more detailed description of them.

In part I and part II we mainly considered problems of the following type:

1.1 Pólya-problems (cf. ref. 22):

The following permutation representation γ_1 of H is considered:

$$\gamma_1 : H \to S_{Y^X} : h \mapsto \begin{pmatrix} \phi \\ \phi \cdot h^{-1} \end{pmatrix}$$

The image of H under γ_1 is denoted by E^H:

$$E^H := \gamma_1[H].$$

It is this type of an enumeration problem which is mainly used in applications to graph theory (cf. 10). If e.g. P denotes a set of points, then we take for X the set $P^{\{2\}}$ of all the unordered pairs of points, put $Y:=\{0,1\}$, and take for H the group $S_P^{\{2\}}$, which is induced from S_P on $P^{\{2\}}$. In this case a $\phi \in Y^X$ may be considered as a graph, an orbit of the group as class of isomorphic graphs (see part I and II).

A generalization of 1.1 is the following type of problem:

<u>1.2</u> <u>de Bruijn-problems (cf. 1)</u>:

We consider the following permutation representation γ_2 of $G\times H$:

$$\gamma_2 : G\times H \to S_{Y^X} : (g,h) \mapsto \begin{pmatrix} \phi \\ g \cdot \phi \cdot h^{-1} \end{pmatrix}$$

The image of $G\times H$ under γ_2 is denoted by G^H:

$$G^H := \gamma_2[G\times H].$$

This group G^H is called the <u>power</u> <u>group</u> of G and H.

This type of problem finds also various applications to graph theory. For example if we again take $X:=P^{\{2\}}$, $Y:=\{0,1\}$, $H:=S_P^{\{2\}}$ and $G:= S_{\{0,1\}}$, then an orbit of the permutation group

$$(S_{\{0,1\}})^{S_P^{\{2\}}}$$

consists of a class of isomorphic graphs with point set P and the class of the complementary graphs. Thus if we denote by B(-) the number of orbits, then

1.3 $$2\cdot B((S_{\{0,1\}})^{S_P^{\{2\}}}) - B(E^{S_P^{\{2\}}})$$

is just the number of classes of selfcomplementary graphs with point set P.

A generalization of 1.2 can be described as follows:

1.4 The exponentiation (8):

Let $G \wr H$ denote the wreath product of G and H, i.e. put

$$G \wr H := \{(f,h) \mid f: X \to G,\ h \in H\},$$

and define the multiplication by

$$(f,h)(f',h') := (ff'_h, hh'),$$

where for each $x \in X$:

$$ff'(x) := f(x)f'(x),\quad f'_h(x) := f'(h^{-1}x).$$

We consider the following permutation representation of this group $G \wr H$:

$$\gamma_3 : G \wr H \to S_{Y^X} : (f,h) \mapsto \binom{\phi}{\psi},\ \text{where } \psi(x) := f(x)\phi(h^{-1}x).$$

The image of $G \wr H$ under γ_3 is called the exponentiation of G and H, and it is denoted by $[G]^H$:

$$[G]^H := \gamma_3[G \wr H].$$

There are only very few applications known of this kind of enumeration problems (e.g. the enumeration of so-called Post-functions), the interested reader may confer ref. 11. But exponentiation groups are sometimes taken for the group H in Pólya-problems, e.g. $H := [S_2]^{S_n}$ arises in the enumeration of Boolean functions.

It is important to notice the following:

1.5 $$E^H \leq G^H \leq [G]^H .$$

In order to make this more precise, we denote by G^*, diagG^*, H' and diag$G^* \cdot H'$ the following subgroups of $G \wr H$:

$$G^* := \{(f,1) \mid f: X \to G\},$$

is called the base group of $G \wr H$ and is a normal subgroup.

$$\text{diag } G^* := \{(f,1) \mid f: X \to G \text{ constant}\}$$

is called the diagonal of the base group. Finally we put

$$H' := \{(e,h) \mid h \in H\},$$

where $e(x) := 1_G$, for all $x \in X$. H' is a complement of G .
Another subgroup is the product

$$\text{diag}G^* \cdot H' = \{(f,h) \mid f: X \to G \text{ constant},\ h \in H\}.$$

We notice that the following holds:

$$G^* \simeq G \times \ldots \times G,\ |X| \text{ factors},$$

$$\operatorname{diag} G^{*} \simeq G,$$
$$H' \simeq H,$$
$$\operatorname{diag} G^{*}\cdot H' \simeq G\times H.$$

Using this notation we obtain a better formulation of 1.5:

<u>1.6</u> (i) $\underline{E^{H} = \gamma_1[H] = \gamma_2[E\times H] = \gamma_3[H'],}$

(ii) $\underline{G^{H} = \gamma_2[G\times H] = \gamma_3[\operatorname{diag} G^{*}\cdot H'].}$

But there is another enumeration problem, which looks quite different. It covers the enumeration of superpositions of graphs, and it was introduced by J.H. Refield in a paper, which was overlooked for many years (ref. 23), but which was published already ten years before the famous paper by Pólya (22) appeared. This problem reads as follows:

<u>1.7</u> <u>Redfield-problems (23)</u>:

Let m and n denote natural numbers and let M be the set of all the (n,m)-matrices, which contain in each of its rows all the elements of $\underline{m} := \{1,\dots,m\}$. We define an equivalence relation "$\sim$" on M as follows:

$$(a_{ik}) \sim (b_{ik}) :\Longleftrightarrow \exists \sigma \in S_m \text{ such that } b_{ik} = a_{i,\sigma^{-1}k}, \text{ all } i,k.$$

Any two such matrices are called <u>column equivalent</u>, since each of them can be obtained from the other one by permuting the columns.

The equivalence class of (a_{ik}) is denoted by $(a_{ik})_\sim$, the set of equivalence classes by $M_\sim$. It is clear that we have for the orders of M and of $M_\sim$:

$$|M| = m!^{n}, \text{ while } |M_\sim| = m!^{n-1}.$$

If now $G_1,\dots,G_n$ denote subgroups of $S_m := S_{\underline{m}}$, then we consider the following permutation representation γ_4 of the direct product $G_1\times\dots\times G_n$:

$$\gamma_4 : G_1 \times\dots\times G_n \to S_{M_\sim} : (g_1,\dots,g_n) \mapsto \begin{pmatrix} (a_{ik})_\sim \\ (g_i a_{ik})_\sim \end{pmatrix}$$

Redfield noticed that there is a bijection between the set of orbits of $\gamma_4[G_1\times\dots\times G_n]$ and the set of certain isomorphism types of superpositions of graphs (cf. 23, 20).

Another case of interest is:

1.8 The matrix group (21):

Let M be defined as in 1.7 and take subgroups $G \leq S_m$ and $H \leq S_n$. We consider the following permutation representation of $G \wr H$:

$$\gamma_5 : G \wr H \to S_{M_\sim} : (f,h) \mapsto \begin{pmatrix} (a_{ik})_\sim \\ (f(i)a_{h^{-1}i,k})_\sim \end{pmatrix} .$$

The image $\gamma_5[G \wr H]$ is called the matrix group of G and H, it is denoted by $[G,H]$:

$$[G,H] := \gamma_5[G \wr H] .$$

In each one of the cases 1.1, 1.2, 1.4, 1.7 and 1.8 the question is how we can get information about the orbits of the permutation groups E^H, G^H, $[G]^H$ and $[G,H]$, assuming a certain knowledge on the orbits of G and H.

The first step will of course be the enumeration of the orbits using Burnside's lemma and the permutation character of the representations γ_i (cf. part II). The last step would be a construction of a complete system of representatives of the orbits. Between these two steps one might derive theorems on the number of orbits with certain properties like prescribed length et cetera. In addition one may ask for a unified treatment of all these results, attempts in this direction have been made in ref. 13 - 16 and 18 - 20.

In order to provide a solution of these problems we shall at first introduce a generalized wreath product of groups together with a particular permutation representation. Afterwards we shall show that this generalizes 1.1, 1.2, 1.4, 1.7 and 1.8. The evaluation of the corresponding permutation character will finally allow a unified treatment of the results, which are hitherto known.

2. A generalization

2.1 Def.: Let H denote a permutation group which acts on $\underline{n}$ and has the orbits $b_1,\ldots,b_r$. Let $G_1,\ldots,G_r$ be groups and put $(G_1,\ldots,G_r)\wr H := \{(f,h) \mid f:\underline{n}\to\bigcup_i G_i, \forall i(i\in b_j \Rightarrow f(i)\in G_j), h\in H\}$. This group is called the generalized wreath product of $G_1,\ldots,G_r$ and H. (The multiplication is defined as in 1.4)

The normal subgroup

$$(G_1,\ldots,G_r)^* := \{(f,1) \in (G_1,\ldots,G_r)\wr H\}$$

is called the base group. If in particular $G_1=\ldots=G_r=G$, then

2.2 $$(G,\ldots,G)\wr H = G\wr H.$$

If now m,n and s denote natural numbers, and if $s \leq m$, then we denote by $M^{<s>}$ the set of all the (n,s)-matrices, which contain in each of their rows just s pairwise different elements of $\underline{m}$. This means that the rows of each $(a_{ik}) \in M^{<s>}$ form injective s-tuples over $\underline{m}$.

For $G_i \leq S_m$, $1 \leq i \leq r$, we define the following permutation representation $\gamma^{<s>}$ of $(G_1,\ldots,G_r)\wr H$ on $M^{<s>}$:

2.3 $$\gamma^{<s>}: (G_1,\ldots,G_r)\wr H \to S_{M^{<s>}}: (f,h) \mapsto \begin{pmatrix} (a_{ik}) \\ (f(i)a_{h^{-1}i,k}) \end{pmatrix}$$

We notice that $M^{<1>}$ may be considered as the set of all the mappings from $\underline{n}$ into $\underline{m}$. This shows that the following is true if we denote by "$\hat{=}$" the similarity of permutation groups:

2.4 (i) $[G]^H = \gamma_3[G\wr H] \hat{=} \gamma^{<1>}[G\wr H]$,

(ii) $G^H = \gamma_3[\mathrm{diag}G^*\cdot H'] \hat{=} \gamma^{<1>}[\mathrm{diag}G^*\cdot H']$,

(iii) $E^H = \gamma_3[H'] \hat{=} \gamma^{<1>}[H']$.

This shows that 2.3 covers Pólya-problems as well as de Bruijn-problems and the exponentiation group. In order to treat Redfield-problems, we introduce the column equivalence of 1.7 as an equivalence relation on $M^{<s>}$. We notice that its classes are just the orbits of the image of the following permutation representation δ of S_s:

2.5 $$\delta: S_s \to S_{M^{<s>}}: \sigma \mapsto \begin{pmatrix} (a_{ik}) \\ (a_{i,\sigma^{-1}k}) \end{pmatrix} .$$

The salient point is the compatibility of δ and $\gamma^{<s>}$ in the following sense:

2.6 Def.: Let α: $P \to S_R$ and β: $Q \to S_R$ denote permutation representations of the groups P and Q on the same set R. We call β α-compatible if and only if the following holds:

$$\forall (p,q,r) \in P\times Q\times R \ \exists q' \in Q \ (\alpha(p)\beta(q)r = \beta(q')\alpha(p)r).$$

Thus β is α-compatible e.g. if $\alpha[P]$ is contained in the normalizer of $\beta[Q]$ in S_R.

If β is α-compatible and if $\overline{R}$ denotes the set of orbits of $\beta[Q]$ on R, $\overline{r}$ the orbit of $r \in R$, then it is easy to check that the following defines a permutation representation $\overline{\alpha}$ of P on $\overline{R}$:

2.7 $$\overline{\alpha}: P \to S_{\overline{R}}: p \to \begin{pmatrix} \overline{r} \\ \overline{\alpha(p)r} \end{pmatrix} .$$

The following result of de Bruijn (2) on the character of $\overline{\alpha}$ is important:

2.8 Lemma: The number of $a_1(\overline{\alpha}(p))$ of points fixed under $\overline{\alpha}(p)$ is equal to

$$\frac{1}{|Q|} \sum_{q\in Q} |\{r \mid r \in R, \ \alpha(p)r = \beta(q)r\}|.$$

It is not difficult to verify that this is true.

We shall apply this to the permutation representations $\gamma^{<s>}$ of $(G_1,\ldots,G_r)\wr H$ and δ of S_s. It turns out that δ is $\gamma^{<s>}$-compatible, so that we obtain by 2.6 and 2.7 the following permutation representation $\overline{\gamma^{<s>}}$ of $(G_1,\ldots,G_r)\wr H$ on $\overline{M^{<s>}}$:

2.9 $$\overline{\gamma^{<s>}}: (G_1,\ldots,G_r)\wr H \to S_{\overline{M^{<s>}}}: (f,h) \to \begin{pmatrix} \overline{(a_{ik})} \\ \overline{(f(i)a_{h^{-1}i,k})} \end{pmatrix} .$$

This representation has the dimension

2.10 $$|\overline{M^{<s>}}| = \binom{m}{s} s!^{n-1} .$$

We denote the image of this permutation group by $[G_1,\ldots,G_r;H]_s$ and call this group a <u>generalized</u> <u>matrix</u> <u>group</u>:

2.11 $$[G_1,\ldots,G_r;H]_s := \overline{\gamma^{<s>}}[(G_1,\ldots,G_r)\wr H] \; .$$

It is clear that in the case when s := m the following holds:

<u>2.12</u> <u>Lemma</u>:

(i) $\underline{\overline{\gamma^{<m>}}[(G_1,\ldots,G_r)\wr\{1_{S_n}\}] = \gamma_4[G_1 \times\ldots\times G_n]}$,

(ii) $\underline{\overline{\gamma^{<m>}}[(G,\ldots,G)\wr H] = \gamma_5[G\wr H]}$.

This together-with 2.4 shows that $\gamma^{<s>}$ and the corresponding representation $\overline{\gamma^{<s>}}$ generalize the enumeration problems, which were introduced in the first section.

<u>3. The permutation characters</u>

The groups $G_1,\ldots,G_r$ were subgroups of S_m acting on $\underline{m}$. Let us denote by $G_i^{[s]}$ the permutation group which is induced by G_i on the set $\underline{m}^{[s]}$ of the injective s-tuples of elements of $\underline{m}$ and by $N_i^{[s]}$ the corresponding natural representation of $G_i^{[s]}$, regarded as representation of G_i, $1 \le i \le r$.

We recall that the orbits of $H \le S_n$ were denoted by $b_1,\ldots,b_r$, and we define a mapping Φ as follows:

3.1 $$\Phi\colon \underline{m} \to \underline{r}\colon i \mapsto j \; , \text{ if } i \in b_j.$$

Then the corresponding outer tensor product

$$\mathop{\#}_{i=1}^{n} N_{\Phi(i)}^{[s]} := N_{\Phi(1)}^{[s]} \# \ldots \# N_{\Phi(n)}^{[s]}$$

is a representation of the base group $(G_1,\ldots,G_r)^*$ of $(G_1,\ldots,G_r)\wr H$. If V_i denotes the representation space of $N_i^{[s]}$, then it operates on $V_{\Phi(1)}\otimes\ldots\otimes V_{\Phi(n)}$ as follows:

$$(f,1)(v_{\Phi(1)}\otimes\ldots\otimes v_{\Phi(n)}) := f(1)v_{\Phi(1)}\otimes\ldots\otimes f(n)v_{\Phi(n)}.$$

The following equation defines an extension of this representation to $(G_1,\ldots,G_r)\wr H$ (cf. 16, 2.3):

$$(f,h)(v_{\phi(1)} \otimes \ldots \otimes v_{\phi(n)}) := f(1)v_{\phi(h^{-1}1)} \otimes \ldots \otimes f(n)v_{\phi(h^{-1}n)}.$$

We denote this extension by

$$\widetilde{\mathop{\#}_{i=1}^{n} N^{[s]}_{\phi(i)}} .$$

It is obvious that we have the following:

3.2 Lemma: $\gamma^{<s>}$ has the same character as $\widetilde{\mathop{\#}_{i=1}^{n} N^{[s]}_{\phi(i)}}$.

We would like to evaluate this character. In order to do this we notice first that the representation $N^{[s]}$ of S_m (on $\underline{m}^{[s]}$) is transitive, so that it is induced from the stabilizer of any injective s-tuple, in particular it is therefore induced from the subgroup S_{m-s}, the stabilizer of the last s symbols. If we denote as usual (cf. 12, 4.7) by $[m-s]$ the identity representation of S_{m-s}, then we have

$$N^{[s]} = [m-s] \uparrow S_m .$$

Denoting by "$\uparrow$" the induction, by "$\downarrow$" the restriction of representations, this yields:

3.3 $$N_i^{[s]} = [m-s] \uparrow S_m \downarrow G_i .$$

In order to use this for an evaluation of the character, we denote the cycle decomposition of $h \in H$ as follows:

$$h = \prod_{\nu=1}^{c(h)} (j_\nu, h\, j_\nu, \ldots, h^{l_\nu - 1} j_\nu)$$

(j_ν being the least symbol in the cyclic factor $(j_\nu \ldots h^{l_\nu -1} j_\nu)$ of h and $j_1 \leq j_2 \leq \ldots < j_{c(h)}$), so that there are uniquely determined elements

$$g_\nu(f,h) := f(j_\nu) f(h^{-1} j_\nu) \ldots f(h^{-l_\nu + 1} j_\nu),$$

which we call the <u>cycleproducts</u> corresponding to the cyclic factors of h with respect to the mapping f. It is known that the cycleproducts determine the conjugacy class of (f,h) in $G \wr S_n$, and hence they show up in formula 2.12 of ref. 16 for the character of certain representations of wreath products. An application of this formula yields the character of $\gamma^{<s>}$ by 3.2:

3.4 Theorem:

If $(f,h) \in (G_1,\dots,G_r)\wr H$ and $g_\nu(f,h)$ are the corresponding cycleproducts of h with respect to f, then we have for the number $a_1(\gamma^{<s>}(f,h))$ of points fixed under $\gamma^{<s>}(f,h)$:

$$a_1(\gamma^{<s>}(f,h)) = \prod_{\nu=1}^{c(h)} \chi^{[m-s]\uparrow S_m}(g_\nu(f,h)) .$$

Putting s:=1, we obtain by an application of 2.4 to 3.4 immediately the theorems of Pólya, de Bruijn and Kerber on the number B(-) of orbits of E^H, G^H and $[G]^H$ as corollaries (put $|X|=n$, $|Y|=m$, and denote by $a_k(h)$ the number of k-cycles in h):

3.5 Corollary:

(i) $B(E^H) = \frac{1}{|H|} \sum_{h\in H} m^{c(h)}$,

(ii) $B(G^H) = \frac{1}{|G||H|} \sum_{(g,h)\in G\times H} \prod_{k=1}^{n} a_1(g^k)^{a_k(h)}$,

(iii) $B([G]^H) = \frac{1}{|G|^n|H|} \sum_{(f,h)\in G\wr H} \prod_{\nu=1}^{c(h)} a_1(g_\nu(f,h))$.

In order to provide an example we evaluate (cf. 1.3) the number

$$2\cdot B(S_2{}^H) - B(E^H)$$

of classes of selfcomplementary mappings with respect to H. 3.5 yields

3.6 $$\frac{1}{|H|} \sum_{h\in H}{}' \ 2^{c(h)} ,$$

where the restricted summation Σ' is to be taken over all $h\in H$ which consist of cyclic factors of even lengths only.

It remains to evaluate the character of $\overline{\gamma^{<s>}}$. We apply lemma 2.8. This lemma shows, that the number of points fixed under $\overline{\gamma^{<s>}}(f,h)$ is equal to the following expression:

3.7 $$\frac{1}{s!} \sum_{\sigma\in S_s} |\{(a_{ik}) \in M^{<s>} \mid \forall\, i,k(a_{i,\sigma^{-1}k} = f(i)a_{h^{-1}i,k})\}|.$$

To make this more explicit, we iterate $a_{i,\sigma^{-1}k}=f(i)a_{h^{-1}i,k}$ so that we obtain

$$\forall\, t \in \mathbb{N},\ i,k(a_{i,\sigma^{-t}k} = f(i)f(h^{-1}i)\ldots f(h^{-t+1}i)a_{h^{-t}i,k}).$$

This yields in particular:

3.8 $\forall\, 1 \leq \nu \leq c(h),\ 1 \leq k \leq s(a_{j_\nu,\sigma^{-1}\nu_k} = g_\nu(f,h)a_{j_\nu,k}).$

Let now $s_{\nu\lambda}$, $1 \leq \lambda \leq c(\sigma^{-1}\nu)$ denote the least symbols in the cyclic factors of $\sigma^{-1}\nu$, $1 \leq \nu \leq c(h)$. 3.8 shows by successive application that the coefficient $a_{j_\nu,s_{\nu\lambda}}$ determines the j_ν-th row of the matrix (a_{ik}) completely. Furthermore 3.8 shows that each $a_{j_\nu,s_{\nu\lambda}}$ occurs in $g_\nu(f,h)$ in a cyclic factor of the same length as $s_{\nu\lambda}$ in $\sigma^{-1}\nu$. And finally $a_{j_\nu,s_{\nu\lambda}}$ and $a_{j_\nu,s_{\nu\lambda'}}$ must occur in different cycles of $g_\nu(f,h)$ as soon as $\lambda \neq \lambda'$ since the rows are injective s-tuples.

This shows that the coefficients $a_{j_\nu,s_{\nu\lambda}}$ uniquely determine the whole matrix (a_{ik}). The number of matrices (a_{ik}) with the property

3.9 $$\forall i,k(a_{i,\sigma^{-1}k} = f(i)a_{h^{-1}i,k})$$

is therefore equal to the product (over all ν, $1\leq\nu\leq c(h)$) of the numbers of injections from $\underline{s}$ into $\underline{m}$, which have the property that the cyclic factors of $\sigma^{-1}\nu$ are mapped onto cyclic factors of $g_\nu(f,h)$. But this number of injections is equal to

$$\prod_{k=1}^{s} \binom{a_k(g_\nu(f,h))}{a_k(\sigma^{-1}\nu)} a_k(\sigma^{-1}\nu)!\,k^{a_k(\sigma^{-1}\nu)} .$$

This altogether proves the following :

<u>3.10</u> <u>Theorem</u>:

<u>The number $a_1(\overline{\gamma^{<s>}}(f,h))$ of points of $\overline{M^{<s>}}$ which remain fixed under $\overline{\gamma^{<s>}}(f,h)$ is equal to</u>

$$\frac{1}{s!} \sum_{\sigma\in S_s} \prod_{\nu=1}^{c(h)} \prod_{k=1}^{s} \binom{a_k(g_\nu(f,h))}{a_k(\sigma^{-1}\nu)} a_k(\sigma^{-1}\nu)!\,k^{a_k(\sigma^{-1}\nu)} .$$

Because of 2.12 we are interested in particular in the case s:=m. We obtain

$$3.11 \quad a_1(\overline{\gamma^{<m>}}(f,1)) = \frac{1}{m!} \sum_{\sigma\varepsilon S_m} \prod_{\nu=1}^{n} \prod_{k=1}^{m} \binom{a_k(f(\nu))}{a_k(\sigma)} a_k(\sigma)!\,k^{a_k(\sigma)} .$$

This yields Redfield's main result since the product in 3.11 does not vanish only if all the $f(\nu)$ are of the same cycle-type as σ, in which case the double product is equal to

$$(\prod_k a_k(\sigma)!\,k^{a_k(\sigma)})^n.$$

Applying this to 3.11, we obtain Redfield's result from 2.12. It reads as follows:

<u>3.12</u> <u>Theorem</u>:

$$a_1(\gamma_4(g_1,\ldots,g_n)) = \begin{cases} (\prod_k a_k!\,k^{a_k})^{n-1}, \text{ if all the } g_i \text{ are} \\ \text{of cycle-type } (a_1,\ldots,a_m) \\ 0, \text{ otherwise} \end{cases}$$

The permutation character of the matrix group turns out to be:

<u>3.13</u> <u>Theorem</u>:

$$a_1(\gamma_5(f,h)) = \frac{1}{m!} \sum_{\sigma\varepsilon S_m} \prod_{\nu=1}^{c(h)} \prod_{k=1}^{m} \binom{a_k(g_\nu(f,h))}{a_k(\sigma^{-l_\nu})} a_k(\sigma^{-l_\nu})!\,k^{a_k(\sigma)} .$$

Applications of the lemma of Burnside yield the various numbers of orbits.

This provides a unified treatment for the derivation of the permutation characters of the enumeration problems introduced in section 1, so that we are now in a position to start a closer examination of the orbits. Klass has shown in ref. 17, how a knowledge of the subgroup lattice of the permutation group in question allows the enumeration of the orbits of prescribed length. But the subgroup lattice is mostly unknown, so that other properties of the orbits should be prescribed.

4. Enumeration of orbits by weight

It is our aim now to obtain generating functions, which enumerate the orbits of Y^X by weight. This means that we are looking for polynomials $p \in \mathbb{Q}[z_1,\ldots,z_{|Y|}]$, where the coefficient of

$$z^\alpha := z_1^{\alpha_1} \ldots z_{|Y|}^{\alpha_{|Y|}}$$

is equal to the number of the orbits, which consist of mappings ϕ with the property

$$\alpha_i = |\phi^{-1}[\{y_i\}]|,\ 1 \leq i \leq |Y|.$$

More generally we can start off with a weight function

4.1 $$w\colon Y \to \mathbb{Q}[z_1,\ldots,z_{|Y|}],$$

which yields the mapping

4.2 $$w^*\colon Y^X \to \mathbb{Q}[z_1,\ldots,z_{|Y|}]\colon \phi \mapsto \prod_{x\in X} w(\phi(x)).$$

We call the functions w , which arise in this way, a multiplicative weight of Y^X.

It is easy to see that multiplicative weights of Y^X are $[G]^H$-compatible, i.e. they are constant on each orbit of $[G]^H$, as soon as w is constant on the orbits of G (on Y). We notice that this implies that such a w^* is furthermore G^H- and E^H-compatible. w^* is E^H- compatible even without any assumption on w.

Using such a weight w^* on Y^X we can define a weight W on the set B of orbits b by

4.3 $$W\colon B \to \mathbb{Q}[z_1,\ldots,z_{|Y|}]\colon b \mapsto w^*(\phi), \text{ for any } \phi \in b.$$

It is our aim to choose w in such a way that the polynomial

4.4 $$\sum_{b\in B} W(b)$$

is the generating function for the enumeration problem in question. Putting w = 1 for example, 4.4 yields just the number of orbits.

We would therefore like to evaluate the polynomial 4.4 for a given weight function w. Let us at first consider this problem for the power group G^H being the permutation group on Y^X, the exponentiation group will be treated later on.

In order to apply representation theoretical arguments again, we consider the corresponding representation

$$\widetilde{\#^{|X|} N} \downarrow \mathrm{diag}G^{*}\cdot H',$$

N being the natural representation of G, acting on the tensor power $\otimes^{|X|}\mathbb{R}^{|Y|}$ as follows:

4.5 $$(g,h)\, v_1 \otimes \ldots \otimes v_{|X|} := gv_{h^{-1}1} \otimes \ldots \otimes gv_{h^{-1}|X|}.$$

The tensor power is therefore a left G-module and a left H-module. Thus (cf. ref. 3,4) the homogeneous component of the identity representation of G affords a representation of H, so that the enumeration problem of the power group can be solved by an application of Burnside's lemma to this particular representation of H. This representation is called the <u>permutrization</u> of the natural representation of G by the identity representation I of G, and it is denoted by

4.6 $$N \triangle_{|X|} I.$$

It has the character (cf. 3,4)

<u>4.7</u> $$\chi^{N \triangle_{|X|} I}(h) = \frac{1}{|G|} \sum_{g\in G} \prod_{k=1}^{|X|} \mathrm{trace}(g^k)^{a_k(h)} .$$

This shows that we obtain from a G-compatible weight w the following expression for 4.4:

4.8 $$\sum_{b\in B} W(b) = \frac{1}{|H|} \sum_{\substack{h\in H\\ g\in G}} \frac{1}{|G|} \prod_{k=1}^{|X|} \Big(\sum_{\substack{i\\ g^k y_i = y_i}} w(y_i)^k \Big)^{a_k(h)} .$$

Immediate corollaries of this are the well known enumeration theorems of Pôlya (22) and de Bruijn (1,9) for E^H and G^H in weighted form:

<u>4.9 Corollary:</u>

(i) <u>If $G \leq S_m$, $H \leq S_n$, $w: \underline{m} \to \mathbb{Q}[z_1,\ldots,z_m]$ is G-compatible, then we have for the orbits $b_1,\ldots,b_r$ of G^H and for $W(b_i) := w^*(\phi)$, $\phi \in b_i$, $w^*(\phi) := \prod_{i=1}^{n} w(\phi(i))$:</u>

$$\sum_{i=1}^{r} W(b_i) = \frac{1}{|G||H|} \sum_{(g,h)\in G\times H} \prod_{k=1}^{n} \Big(\sum_{\substack{j=1 \\ g^k j = j}}^{m} (w(j)^k) \Big)^{a_k(h)} .$$

(ii) <u>If $H \leq S_n$, $w: \underline{m} \to \mathbb{Q}[z_1,\ldots,z_n]$ any weight function, then we have for the orbits $b_1,\ldots,b_s$ of E^H and for $W(b_i) := w^*(\phi) := \prod_{i=1}^{n} w(\phi(i))$, $\phi \in b_i$:</u>

$$\sum_{i=1}^{s} W(b_i) = \frac{1}{|H|} \sum_{h\in H} \prod_{k=1}^{n} \Big(\sum_{j=1}^{m} w(j)^k \Big)^{a_k(h)} .$$

In order to derive an enumeration theorem for the exponentiation group in weighted form, one uses the weighted form of Burnside's lemma and considers the trace of a certain operator on a tensor space.

The notation, which is needed in order to describe this, is quite complicated, but not difficult, we therefore prefer to leave it out. The interested reader is referred to references 19, 20 and 15.

It is clear that the enumeration theorem of the exponentiation group in weighted form implies the theorems of the power group and of the group E^H, since both these are subgroups of the exponentiation group. One may therefore derive the exponentiation group theorem first and then get 4.9 as corollary. We preferred to do it the other way round since we wanted to point to the representation theoretical concept of permutrization which gives 4.9 directly.

It should be mentioned that in a sense dual to the permutrization is the concept of symmetrization, which is used in ref. 14. The theorem on the exponentiation group reads:

<u>4.10</u> <u>Theorem</u>:

<u>If $G \leq S_m$, $H \leq S_n$, $w: \underline{m} \to \mathbb{Q}[z_1,\ldots,z_m]$ is G-compatible, then we have for the orbits $b_1,\ldots,b_t$ of $[G]^H$ and for $W(b_i) := w^*(\phi) := \prod_{i=1}^{n} w(\phi(i))$:</u>

$$\sum_{i=1}^{t} W(b_i) = \frac{1}{|G|^n |H|} \sum_{(f,h)\in G\wr H} \prod_{\nu=1}^{c(h)} \Big(\sum_{g_\nu(f,h)j=j} w(j)^{l_\nu} \Big).$$

5. Cycle-indices

It is well known (see parts I and II e.g.) that the enumeration theorems in weighted form can be formulated in terms of certain polynomials $Z(H)$, the so-called cycle-indices, which are defined as follows:

5.1 $$Z(H) := \frac{1}{|H|} \sum_{h\in H} \prod_{k=1}^{|X|} z_i^{a_i(h)} \in \mathbb{Q}[z_1,\ldots,z_{|X|}].$$

4.9 for example shows that the generating function for the Pólya-problem can be obtained from $Z(h)$ by substituting $\sum_j w(j)^k$ for z_k in $Z(H)$. We express this by

5.2 $$\sum_{i=1}^{s} W(b_i) = Z(H \mid \sum_j w(j)).$$

We call this substitution Pólya-substitution.

Using this notation 5.2, we obtain result 4.8 in the following form:

5.3 Theorem:

Under suitable assumptions (cf. 4.9) we have

$$\sum_{i=1}^{s} W(b_i) = Z(H \mid \sum_j w(j)),$$

The evaluation of the polynomial $Z(H)$ is therefore of great interest. We would like to draw the attention of the reader to a general method, which allows to evaluate the cycle-index from the permutation character, subject to the condition that we know for each element of the permutation group in question, to which conjugacy classes of the group its powers belong.

In order to describe this, we denote by $\alpha: P \to S_R$ a permutation representation of the group P. We assume that for each $p \in P$ and every $r \in \mathbb{N}$ the conjugacy class of p^r in P is known. Then the cycle-index of $\alpha[P]$ can be evaluated from the permutation character of α since we have for the number $a_i(\alpha(p))$ of i-cycles in $\alpha(p)$:

5.4 $$a_i(\alpha(p)) = \frac{1}{i} \sum_{k|i} \mu(\frac{i}{k}) a_1(\alpha(p)^k) .$$

This follows by an application of the Moebius-inversion to

5.5 $$a_1(\alpha(p)^k) = \sum_{i|k} i \cdot a_i(\alpha(p)) .$$

This can be applied to 3.4 and 3.10 and yields the cycle-indices of $\gamma^{<s>}[(G_1,\ldots,G_r)\wr H]$ as well as $\gamma^{<s>}[(G_1,\ldots,G_r)\wr H]$. Corollaries are the known theorems on the cycle-indices of E^H, G^H and $[G]^H$. The resulting expressions for the cycle-indices are awfully complicated, but with the exception of trivial cases such cycle-indices can be evaluated anyway only with the aid of a computer. This method of Moebius-inversion provides also the cycle-indices of $[G;H]$ and $[G_1,\ldots,G_r;H]_s$.

It should be remarked that in the case of the weight function

5,6 $$w: Y \to \mathbb{Q}[z_1,\ldots] : y_i \mapsto z_i,$$

the Pólya-substitution yields a polynomial, which already Redfield had used, and which is called the group reduction function:

5.7 $$\mathrm{Grf}\,(H) := Z(H \mid \sum_j z_j) = \frac{1}{|H|} \sum_h \prod_{k=1}^{|X|} (z_1{}^k + \ldots + z_{|Y|}{}^k)^{a_k(h)} .$$

Foulkes pointed to the close connection between Grf(H) and the Schur-functions (ref. 5,6). We described this already in part II.

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