match no. 3 pp. 51 - 66 1977

On Computing Plethysms of Schur-functions

P.A. Morris and G.H.J. van Rees
(received: July 1976)

Introduction

The plethysm of Schur-functions (S-functions) was defined by Littlewood in [3]. Since then several papers (see [10], [12] for references) have appeared on the expansion of the plethysm in terms of S-functions. Recently there has been renewed interest in the problem since the appearance of the expository paper by Read [10] describing the application of S-functions to combinatorial analysis, and of the papers of Smith and Wybourne et al ([1], [11]) on the application of the plethysm to the theory of complex spectra. A summary of Littlewood's work is well presented in the book by Wybourne[12].

In this paper we outline methods derived in an earlier paper [7] by one of us and then we discuss algorithms for a certain set of plethysms. These algorithms have been used to produce the plethysms $h_m[h_3]$ ($\{3\} \otimes \{m\}$ in Littlewood's notation) for $m \leq 12$. The tables give a variety of combinatorial 'numbers', notably the number of cubical graphs on 12 or fewer nodes.

2. Definitions

Corresponding to each partition (λ) of an integer n there exists a character $\chi_\rho^{(\lambda)}$ of the symmetric group S_n for each partition $(\rho) = (1^{j_1} 2^{j_2} \dots n^{j_n})$ of n.

If $s_\rho = s_1^{j_1} s_2^{j_2} \dots s_n^{j_n}$ where s_r is the power-sum symmetric function $\Sigma\alpha_i^r$ (the α_i are indeterminates) then the S-function $\{\lambda\}$ is a symmetric function defined by

$$\{\lambda\} \equiv \{\lambda_1, \lambda_2, \dots, \lambda_n\} = \sum_\rho \chi_\rho^{(\lambda)} s_\rho \quad .$$

If h_r is the homogeneous power-sum symmetric function

$\Sigma\alpha_1^{r_1}, \alpha_2^{r_2} \ldots \alpha_n^{r_n}$, where the sum is over all $r_1, r_2, \ldots, r_n$ such that $r_1 + r_2 + \ldots + r_n = r$ then it is known that (see, for example [5])

$$\{\lambda\} = |h_{\lambda_1 - s + t}| = \begin{vmatrix} h_{\lambda_1} & h_{\lambda_1+1} & h_{\lambda_1+2} \cdots h_{\lambda_1+n-1} \\ h_{\lambda_2-1} & h_{\lambda_2} & h_{\lambda_2+1} \cdots h_{\lambda_2+n-2} \\ \vdots & & \ddots \\ h_{\lambda_n-n+1} & \cdot & \cdots \; h_{\lambda_n} \end{vmatrix} \qquad (2.1)$$

In particular, the one-part S-function $\{n\}$ is equal to h_n (by either definition) so that h_{λ_1} is simply $\{\lambda_1\}$.

(We shall use a convention adopted in [10]: that a Roman letter or subscripted Greek letter will stand for an integer, and a non-subscripted Greek letter for a partition).

We shall be concerned with two products of S-functions: the ordinary, commutative (inner) product, for which there is an effective algorithm [6], and the plethysm, of which we give one of the several equivalent definitions.

If A and B are polynomials in the power-sum symmetric functions $s_1, s_2, \ldots, s_m$ then by 'substitution of B into A' we mean the replacing of each s_r in A by the polynomial obtained from B by multiplying by r the subscript of each of the s's. For example, if

$$A = \frac{1}{2}(s_1^2 + s_2), \qquad B = \frac{1}{3}(2s_1 + s_3^2)$$

then we perform the substitution by replacing

$$s_1 \text{ by } \frac{1}{3}(2s_1 + s_3^2)$$

$$s_2 \text{ by } \frac{1}{3}(2s_2 + s_6^2)$$

to give $$\frac{1}{2}\{\frac{1}{9}(2s_1 + s_3^2)^2 + \frac{1}{3}(2s_2 + s_6^2)\}$$

The result is denoted by $A[B]$.

If we write the S-functions $\{\lambda\}$ and $\{\mu\}$ in terms of the s_r functions, the 'substitution' $\{\lambda\}[\{\mu\}]$ is a polynomial in the power-sum symmetric functions, and hence can be rewritten as a sum of S-functions. The equation $\{\lambda\}[\{\mu\}]= \sum_{\nu} D_{\nu}\{\nu\}$, where D_{ν} is the coefficient of $\{\nu\}$, defines the plethysm. In Littlewood's notation, $\{\lambda\}[\{\mu\}]= \{\mu\} \otimes \{\lambda\}$. D_{ν} is at least zero, and if (λ) is a partition of m, and $\{\mu\}$ a partition of n, D_{ν} is non-zero only if (ν) is a partition of mn. For ease of typing we will normally use $N(\{\nu\}*$ instead of D_{ν}. Thus the coefficient of ν in $h_m[h_n]$ is $N(\{\nu\} * h_m[h_n])$, and also

$$N(\{\nu\} *\{\lambda\}) = 1 \text{ if } (\nu) = (\lambda)$$
$$= 0 \text{ otherwise,}$$

(ν) and (λ) being both partitions of n.

We shall be concerned here with the plethysm $\{m\} [\{n\}]= h_m[h_n]$, i.e. the case in which $\{\lambda\}$ and $\{\mu\}$ are both one-part S-functions.

3. <u>The Plethysm $\{3\} \otimes \{m\}$</u>

In [7] we derived a recursive formula for $D_{\lambda}\{3\} \otimes \{m\}$, where $\{\lambda\} \equiv \{\lambda_1, \lambda_2,\ldots,\lambda_m\}$ is an S-function with $\lambda_1 > m$. In order to show how we apply the formula, and how we find the coefficients of those S-functions not covered by the restriction $\lambda_1 > m$, we first state the following theorem (Read [9]) without proof.

<u>Theorem 1</u>. The number of non-isomorphic bipartite graphs with m unlabelled nodes in a subset A_1 of the nodes, and nodes of valency $\lambda_1, \lambda_2,\ldots, \lambda_m$ in the other subset A_2, is

$$N((\lambda_1)(\lambda_2)\ldots(\lambda_m)* h_m[h_n]).$$

Now let $G_{\rho}\{\lambda\}$ $h_m[h_n]$ denote the operator

$\Sigma\, N(\{\lambda_1 - \rho_1, \lambda_2 - \rho_2, \lambda_3 - \rho_3 .. \lambda_m - \rho_m\}^* h_{m-r}[h_n])$ where the summation is over all permutationsof $\rho_2, \rho_3, \ldots, \rho_m$, and (ρ) $(\rho_1, \rho_2, \ldots, \rho_m)$ is a partition of nr. Thus, if $\{\lambda\} = \{4, 4, 1\}$ then

$G_{2,1}\{\lambda\}\, h_3[h_3] = N(\{2, 3, 1\} + \{2, 4, 0\}^* h_2[h_3])$.

In applying this operator we shall often need the following properties of S-functions

a) $\{\lambda_1, \lambda_2, \ldots, \lambda_i, \lambda_{i+1}, \ldots, \lambda_m\} = -\{\lambda_1, \lambda_2, \ldots, \lambda_{i+1}-1, \lambda_{i-1}+1, \ldots, \lambda_m\}$

Thus $\{2, 4, 0\} = -\{3, 3, 0\}$

and $\{2, 3, 1\} = -\{2, 3, 1\}$ (whence $\{2, 3, 1\}$ vanishes identically).

b) If the last non-zero part of $\{\lambda\}$ is negative, $\{\lambda\}$ is identically zero.

<u>Theorem 2</u>. If $\lambda_1 > m$ then

$$N(\{\lambda_1, \lambda_2, \ldots, \lambda_m\}^* h_m[h_3])$$
$$= [G_3 + G_{2,1} - G_3G_{2,1} - G_{2,1}^2 + G_3G_{2,1}^2 + G_{2,1}^3 - \ldots]h_m[h_3]$$

This is proved in [7]. We will give a brief outline of the proof here, since similar ideas will be used in deriving the other two theorems which we need. First, as an example of the use of this theorem

$$\begin{aligned} N(\{4,4,1\}^* h_3[h_3] &= N(\{4 - 3, 4, 1\}^* h_2[h_3]) \\ &+ N(\{4 - 2, 4 - 1, 1\} + \{4 - 2, 4, 1 - 1\}^* h_2[h_3] \\ &- N(\{4 - 5, 4 - 1, 1\} + \{4 - 5, 4, 1 - 1\}^* h_1[h_3] \\ &- N(\{4 - 4, 4 - 1, 1 - 1\}^* h_1[h_3] \\ &+ N(\{4 - 7, 4 - 1, 1 - 1\}^* h_0[h_3] \\ &+ \ldots\ldots\ldots\ldots \end{aligned}$$

$= N(\{1, 4, 1\} + \{2, 3, 1\} + \{2, 4\} * h_2[h_3])$

$- N(\{-1, 3, 1\} + \{-1, 4\} + \{0, 3\} * h_1[h_3])$

$- N(\{-3\ 3\} + \{-2, 3, 0, -1\} * h_0[h_3])$

$= N(\{3, 2, 1\} - \{3, 3\} * h_2\ h_3 * h_2[h_3]) - N(-\{3\} + \{2, 1\} * h_1[h_3])$

(applying properties (a) and (b)).

$= 1.$

(It is only necessary to go as far as $h_0[h_3] = \{0\}$).

Consider a bipartite graph with the nodes in two subsets: A consisting of m unlabelled nodes each of valency 3 and B consisting of nodes with valencies λ_1, $\lambda_2, \ldots, \lambda_m$ with $\lambda_1 > m$.

Since $\lambda_1 > m$, the node with this valency cannot be connected to nodes in A_1 by single edges only, and hence every bipartite graph of the above description contains one or both of the features in figure 1.

Figure 1.

Removing feature (a) in figure 1 gives a bipartite graph with m - 1 unlabelled nodes in A_1, and nodes of valency $\lambda_1 - 3$, $\lambda_2, \ldots, \lambda_m$ in A_2, so that there is an obvious 1 - 1 correspondence between these new graphs and those of the original type which contained feature (a).

By theorem 1, the original ones are counted by
$N(\{\lambda_1\}\ \{\lambda_2\}, \ldots, \{\lambda_m\} * h_m[h_3])$ and these new ones by
$N(\{\lambda_1 - 3\}\{\lambda_2\}, \ldots, \{\lambda_m\} * h_{m-}[h_3])$.

Simarly removal of feature (b) gives the count
$N(\{\lambda_1 - 2\}\ \{\lambda_2 - 1\}\ \{\lambda_3\}, \ldots, \{\lambda_m\} + \{\lambda_1 - 2\}\ \{\lambda_2\}\{\lambda_3 - 1\}, \ldots, \{\lambda_m\} +$
$\ldots + \{\lambda_1 - 2\}\ \{\lambda_2\}, \ldots, \{\lambda_m - 1\} * h_{m-1}[h_3])$.

Application of the principle of inclusion and exclusion to features (a) and (b) gives a result similar to theorem (2), with $N(\{\lambda_1\}\{\lambda_2\}, \ldots, \{\lambda_m\} * h_m[h_3])$ instead of $N(\{\lambda_1, \lambda_2, \ldots, \lambda_m\} * h_m[h_3])$, and this final step is completed by noting that the result holds for each term

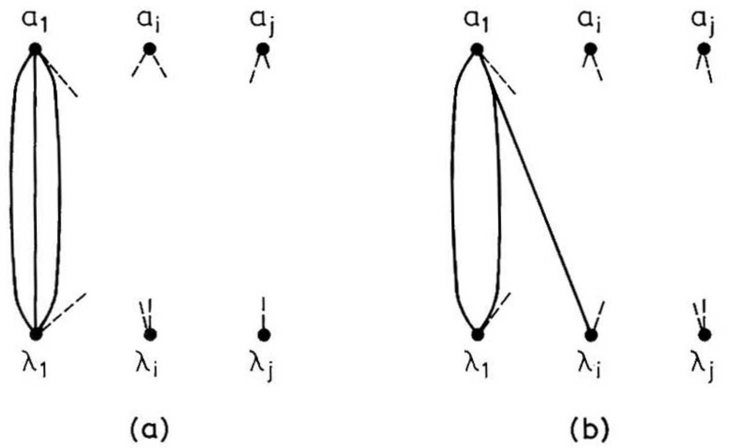

Figure 1

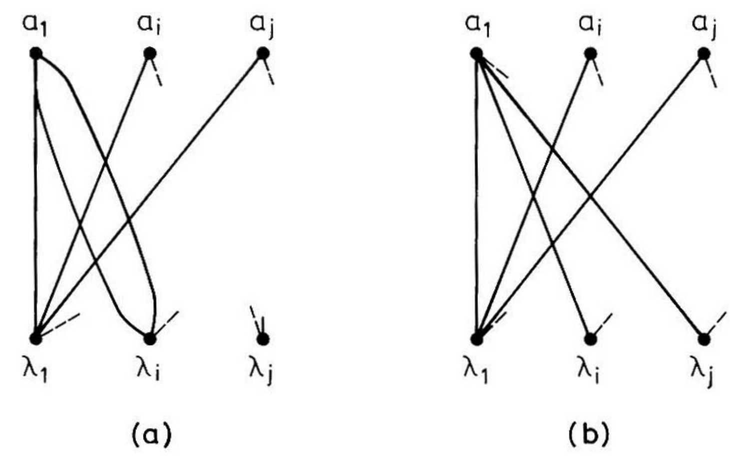

Figure 2

in the expansion of the determinant (2.1).

The algorithm suggested by theorem 2 is very easy to program. Its disadvantage is that it is recursive. It is not possible to obtain a non-recursive algorithm from the theorem as it stands because it is not always true that intermediate S-functions generated by the 'G' operators obey the restriction of the theorem.

In case $\lambda_1 \leq m$ it is no longer true that the only 'features' which can occur are (a) and (b) of figure 1. It is now possible for λ_1 to be connected by single edges only to the nodes in A_1, and the two possibilities are shown as (c) and (d) in figure 2.

Figure 2.

These can be treated in the same manner as features (a) and (b) in theorem (2) and they give rise to terms $G_{1,2}\{\lambda\}\ h_m[h_3]$ and $G_{1,1,1}\{\lambda\}h_m[h_3]$.

A straightforward application of inclusion-exclusion gives a general formula, but the number of terms involved is much larger, and the simplicity of the algorithm is destroyed.

4. <u>Correction for $\lambda_1 \leq m$.</u>

In [7] we also obtained the following.

<u>Theorem 4</u>

$$N(\{\lambda_1\}\ \{\nu\}*\ h_m[h_n]) = \sum_{r=0} \sum_{\mu} N(\{\lambda_1 - m + r\}\{\mu\} *\ h_{m-r}[h_{n-1}])$$

$$N(\{\nu/\mu\}*\ h_r[h_n])$$

where $\{\nu/\mu\}$ is the isobaric determinant $|h_{\nu_s - \mu_t - s + t}|$ ([5], page 109).

Again we consider the same bipartite graph, and suppose that the node of valency λ_1 is connected to $m - r$ unlabelled nodes in A. We can split the graph into two parts. One part G_1 is a bipartite graph with a set A_1 of m - r unlabelled nodes each of valency 3 (n in the general case) and a set B_1 of nodes of valency $\lambda_1, \lambda_2 - \rho_2, \lambda_3 - \rho_3, \ldots, \lambda_m - \rho_m$. The other part G_2 is a bipartite graph with a set A_2 of r unlabelled nodes of valency 3 and nodes of valency $\rho_2, \rho_3, \ldots, \rho_m$, where $\rho_2 + \rho_3, \ldots, + \rho_m = 3r$.

If we remove one edge from each member of A_1 we can form a 1 - 1 correspondence between the first subgraph G_1 and a bipartite graph G_3 with m - r unlabelled nodes each of valency n - 1 (2 in our case). The new set B_1 contains nodes of valency $\lambda_1 - m + r, \lambda_2 - \rho_2, \ldots, \lambda_m - \rho_m$.

The rest of the work in theorem 4 consists in enumerating G_3 and G_2 together.

We are now concerned with the special case in which the original bipartite graph has no multiple edges. It can be shown that in this case the number turns out to be

$$\sum_{k=0}^{m-\lambda_1} (-1)^k \sum_{\mu} N(\{\lambda\}^* h_{\lambda_1+k}[h_2])N(\{\alpha/\mu\}^* h_{m-\lambda_1-k}[h_3]) \tag{4.1}$$

where
$$\{\alpha\} = \{(\lambda_2, \lambda_3, \ldots, \lambda_m)/(1^k)\} .$$

Now it is known that $N(\{\lambda\}^* h_m [h_2]) = 1$ if (λ) is a partition of 2m into even parts, and is zero otherwise [10].

Hence 4.1 reduces to

$$\sum_{k=0}^{m-\lambda_1} (-1)^k \sum_{(2\mu)} N(\{\alpha/2\mu\}^* h_{m-\lambda_1-k}[h_3]) \tag{4.2}$$

$$= \sum_{k=0}^{m-\lambda_1} (-1)^k \sum_{(2\mu)} N(\{\alpha\}^*\{2\mu\}\, h_{m-\lambda_1-k}[h_3])$$

where (2μ) denotes a partition of $2(\lambda_1 + k)$ into even parts only.

Since (4.2) corresponds to the cases in which the node of valency λ_1

has no multiple edges, the expression gives us a correction term when we apply theorem 2 in the case $\lambda_1 \leq m$. Our strategy is thus to calculate this term as a correction to be added to the algorithm based on theorem 2.

Now from [5], pg. 109

$$\{\lambda/\mu\} = \sum_{\alpha}[N(\{\lambda\} * \{\mu\} \{\alpha\})] \quad \{\alpha\}$$

so that $$\{\nu/(1^k)\} = \sum_{\alpha} [N(\{\nu\}*\{1k\} \{\alpha\})] \quad \{\alpha\}.$$

If we apply the well-known algorithm for the ordinary product (see [6]) it is easy to see that the coefficient of $\{\alpha\}$ is 1 or 0. The product $\{1^k\} \{\alpha\}$ is formed by adding in order, reading from right to left and top to bottom, k different new symbols to the Young diagram for $\{\nu\}$. Hence we must be able to form the Young diagram for $\{\alpha\}$ by substracting the new symbols from the diagram for $\{\nu\}$. Hence we select k of the ν_i and substract 1 from each of them. This gives an $\{\alpha\}$ (with coefficient 1), unless $\nu_i < \nu_{i+1}$, in which case we reject this selection of k of the ν_i and go on to the next.

For example, $\{(4, 4, 2, 1)/(1, 1)\} = \{4, 3, 1, 1\} + \{4, 3, 2\} + \{4, 4, 1\}$.

Having obtained these 'target' functions $\{\alpha\}$, for each one we find next $N(\{\alpha\} *\{2\mu\} \; h_{m-\lambda_1-k} [h_3])$. This, of course, requires an algorithm for S-function multiplication. Such an algorithm is described in [6], and we have used a somewhat improved version in the present application.

When $m - \lambda_1 - k = 0$, $h_{m-\lambda_1-k}[h_3]$ is the single S-function $\{0\}$, and the contribution to the correction term is simplified.

We seek $N(\{\alpha\}*\{2\mu\}\{0\}) = N(\{\alpha\}*\{2\mu\}) = 1$ or 0 according as $\alpha \equiv 2\mu$ or not.

Hence in this case the term is 1 or 0 according as (α) consists of even parts only, or not.

In the special case in which $\lambda_1 = m$, the only value of k which can be used is $k = 0$, and hence the whole correction term is 1 if $\lambda_2, \lambda_3, \lambda_4, \ldots, \lambda_m$ are all even, and is zero otherwise.

This second algorithm does, of course, take more time than the first, and much depends on the speed with which we can do two things: generate partitions and multiply S-functions. Fortunately algorithms for both of these are quite fast. In addition this correction term applies only to those S-functions with $\lambda_1 \leq m$, and we have to use S-function tables only as far as $m - \lambda_1 - k$. Also when k is small, so that $m - \lambda_1$ is large, $\lambda_1 + k$ is small so that the number of partitions $\{2\mu\}$ is small, and the number of functions $\{\alpha\}$ is small.

5. Algorithms.

We can now describe the complete process for the plethysm $\{3\} \otimes \{m\}$. As we have seen, the S-functions which occur in the expansion correspond to partitions of $3m$ into at most m parts. Hence we produce one of these partitions at a time, and apply the algorithm to the partition.

If m is less than λ_1, then we need only the algorithm from theorem 2. Applying the operation $G_3 G_{2,1}^2$, for example, means substracting from $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_m$ each of $(5, 1, 1, 0, \ldots, 0)$, $(5, 1, 0, 1, \ldots, 0)$, $(5, 1, 0, 0, 1 \ldots)$, $(5, 1, 0, \ldots, 1)$, $(5, 0, 1, 1, \ldots, 0)$, ... $(5, 0, 0, \ldots, 1, 1)$. There are $2^{m-1} - 1$ of these 'vectors', and one can either create and store them once at the beginning, or generate them on demand. In either case if the last '1' in one of these vectors is in position k, then the result of operating on $(\lambda_1, \lambda_2, \ldots, \lambda_j, 0, 0, 0 \ldots)$ is 0 if $j < k$, since the last non-zero part of the result is negative. Thus it is useful to include the 'length' k of each function.

We can summarize as follows.

Step 1. Generate a partition of $3m$ into a most m parts.

Step 2. Subtract, corresponding to the operator $G_3^r G_{2,1}^s$, $r = 0$ or 1, the vector $(3r + 2s, 1, 0, 1,\dots)$ (with s 1's). Associated with the vector is a sign $(-1)^{r+s+1}$. Call the result $(\mu)' \equiv (\mu_1', \mu_2', \dots, \mu_m')$.

Step 3. If the last non-zero part of $(\mu)'$ is negative, or if, for any i. $\mu_i' + 1 = \mu_{i+1}'$, count zero. Otherwise reduce $\{\mu\}'$ to standard form $\{\mu\} \equiv \mu_1, \mu_2, \dots, \mu_m\}$, with $\mu_1 \geq \mu_2 \geq \dots \geq \mu_m$ by applying property (a) section 3, as many times as necessary, noting the accompanying sign changes.

Step 4. Search the expansion $h_{m-r-s}[h_3]$ for $\{\mu\}$. Count its coefficient multiplied by the sign obtained in Step 3.

Go to step 2 for another operator. If there are no more, store this partition, if its total count is non-zero, together with its coefficient (the count). Go to step 1 for the next partition.

We do not yet know how to estimate beforehand the amount of storage space required for each expansion. An upper bound can obviously be calculated from the number of partitions of $3m$ into at most m parts (which can be found from tables of partitions), multiplied by the 'average' length of a partition, but the result is unsatisfactorily large, since many partitions do not occur: i.e. occur with zero coefficient.

Now in case $\lambda_1 < m$, we have to add the correction term (4.2). We summarize:

Initially set $k = 0$, $(\nu) = (\lambda_2, \lambda_3, \dots, \lambda_m)$.

Step 1. Aubtract 1 from each of k elements of (ν) to give $(\alpha) = (\alpha_1, \alpha_2, \dots, \alpha_{m-1})$, with $\alpha_1 \geq \alpha_2, \dots, \geq \alpha_{m-1}$. If the inequalities do not hold make another selection of k elements.

Step 2. Find a partition of $\lambda_1 + k$ into at most $m-1$ parts. Multiply each part by 2, and denote the result by (2μ).

Step 3. Find the coefficient of $\{\alpha\}$ in $\{2\mu\}\, h_{m-\lambda_1-k}[h_3]$: for each S-function $\{\beta\}$ in $h_{m-\lambda_1-k}\,[h_3]$, find the number of times $\{\alpha\}$ occurs in $\{2\mu\}\{\beta\}$, and multiply the result by the coefficient of $\{\beta\}$ in $h_{m-\lambda_1-k}\left[h_3\right]$.

Step 4. Go to Step 2 and choose another partition. If there are no more partitions add to a counter $(-1)^k$ times the sum of the results of Step 3.

Increment k and go to Step 1.

Step 5. Finish when $m - \lambda_1 - k$ is negative: i.e. when $k > m - \lambda_1$. The total is the correction term to be added to the results of the first algorithm.

6. Conclusion.

The complete procedure, even with the slower correction term, is quite effective. Both algorithms can, of course, be easily generalized. For the first algorithm we have

Theorem 2B: If $\lambda_1 > (n-2)m$ then

$$N(\{\lambda\}* h_m[h_n]) = (G_n + G_{n-1,1} - G_n\, G_{n-1,1} + G^2_{n-1,1} + \dots\,)\ h_m[h_n].$$

Applying this presents no more difficulty than applying theorem 2.

The second algorithm does not follow through as easily. One reason is, of course, that the cases to be enumerated are not simply those in which the node of valency λ_1 is connected by single edges. In the case $n = 4$, for example, theorem 2B omits those cases in which this node is connected by 2 edges or by single edges.

Secondly, the plethysm $h_m[h_2]$ is especially simple, and the ease of calculating it (partitions into even parts)does not carry over to

$h_m[h_3]$ or higher plethysms.

Checking tables produced by these algorithms is not a particularly easy task. In practice the fact that the algorithm is recursive helps: since the coefficients in each expansion are never negative, the lack of negative signs in the higher values is in itself a source of confidence. (In fact, negative signs appear in the early 'debugging' stages).

A number of other theorems have been used. We will merely quote these, with references, not necessarily in their original general forms, but in the special cases which we have used.

1) If $\lambda_m > 0$,

$$N(\{\lambda\}* h_m[h_3]) = N(\{\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_m - 1\}* \{1^m\}[h_2])$$

([2], theorem 28)

and since (by [5]), the coefficient of an S-function in $\{1^m\}[h_2]$ is 1 or 0 according as its Frobenius 10 form is one of the types $\binom{a+1}{a}$, $\binom{a+1,\ b+1}{a\ ,\ \ b}$ etc., this is easy to check.

2) For appropriate functions, let $\{2n - \overline{\lambda}\}$ denote $\{2n - \lambda_m, 2n - \lambda_{m-1}, \ldots, 2n - \lambda_1\}$ (In our case, $n = 3$).

Then $N(\{2n - \overline{\lambda}\}* h_m[h_n]) = N(\{\lambda\}* h_m[h_n])$ ([2], theorem 33).

3) If $\lambda_2 \leq n - 1$ then

$$N(\{\lambda\}* h_m[h_n]) = N(\{\lambda_1 - 3, \lambda_2, \ldots, \lambda_m\}* h_m[h_{n-1}])$$

([2], Theorem 29).

4) If $\lambda_1 > (m-1)(n-1)$

$$N(\{\lambda\}* h_m[h_n]) = N(\{\lambda_1 - m, \lambda_2, \ldots, \lambda_m\}* h_m[h_{n-1}])$$

$$+ N(\{\lambda_1 - n, \lambda_2, \ldots, \lambda_m\} * h_{m-1}[h_n])$$

$$- N(\{\lambda_1 - m - n + 1, \lambda_2, \ldots, \lambda_m\}* h_{m-1}[h_{n-1}])$$ (see [7])

In addition tables have been compiled for $m \leq 6$ by Ibrahim [3] (Royal Society depository of unpublished tables) and these provide an excellent check.

References

1. Butler, P.H., Wybourne, B.G.; Tables of Outer S-Function Plethysms, Atomic Data v.s, 1971, pp. 133-151.
2. Foulkes, H.O.; Plethysm of S-Functions, Trans. Royal Soc. (Lond) A246, 1954, pp. 555-593
3. Ibrahim, E.M.; On D.E. Littlewood's Algebra of S-Functions, Proc. Amer. Math. Soc. v.7, 1956, pp.199-202
4. Littlewood, D.E.; On Induced and Compound Matrices, Proc. London Math. Soc. (2), v.40, 1936, pp. 370-381.
5. Littlewood, D.E.; The Theory of Group Characters, Oxford, 1950.
6. Morris, P.A.; Computational Problems in S-Function Theory, Math. Computation, v. 27, 1973, pp. 965-971.
7. Morris, P.A.; Applications of Graph Theory to S-Function Theory, J. London Math. Soc. (2), v.8, 1974, pp, 63-72.
8. Plunkett, S.P.O.; On the Plethysm of S-Functions, Can. J. Math., v.24, 1972, pp. 541-552.
9. Read, R.C.; The Enumeration of Locally Restricted Graphs, (I).J. London Math. Soc. v.34, 1959, 417-436.
10. Read, R.C.; The Use of S-Functions in Combinatorial Analysis, Canad. J. Math. v.20, 1968, pp. 808-841.
11. Smith, P.R., Wybourne, B.G.; Plethysm and the Theory of Atomic Spectra J. Math. Physics v.9, 1968, pp.1040-1051.
12. Wybourne, B.G.; Symmetry Principles and Atomic Spectroscopy, Wiley 1970.

On Computing Plethysms of Schur Functions

Authors' Addresses

P.A. Morris, Department of Mathematics, University of the West Indies, St. Augustine, Trinidad.

G.H.J. van Rees, Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Ontario, Canada.

Footnote: "This paper was written while the first author was on a Fellowship at the University of Waterloo"

AMS Classification AMS(MOS) (1970), 20C30 Secondary 20-40.

Key Words and Phrases

S-Function
Plethsym
Group Characters
Wreath-Product
Full Linear Group
Schur Function
Young diagram
Symmetric Function